

UNBOUNDED DERIVATIONS, FREE DILATIONS AND INDECOMPOSABILITY RESULTS FOR II_1 FACTORS

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ABSTRACT. We give sufficient conditions, in terms of the existence of unbounded derivations satisfying certain properties, which ensure that a II_1 factor M is prime or has at most one Cartan subalgebra. For instance, we prove that if there exists a real closable unbounded densely defined derivation $\delta : M \rightarrow L^2(M) \bar{\otimes} L^2(M)$ whose domain contains a non-amenability set, then M is prime. If δ is moreover “algebraic” (i.e. its domain M_0 is finitely generated, $\delta(M_0) \subset M_0 \bar{\otimes} M_0$ and $\delta^*(1 \otimes 1) \in M_0$), then we show that M has no Cartan subalgebra. We also give several applications to examples from free probability. Finally, we provide a class of countable groups Γ , defined through the existence of an unbounded cocycle $b : \Gamma \rightarrow \mathbb{C}(\Gamma/\Lambda)$, for some subgroup $\Lambda < \Gamma$, such that the II_1 factor $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra, up to unitary conjugacy, for any free ergodic probability measure preserving (pmp) action $\Gamma \curvearrowright (X, \mu)$.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Background. A central theme in the theory of von Neumann algebras is to investigate various decompositions of II_1 factors, such as tensor product and Cartan decompositions. Recall that a II_1 factor M is *prime* if it cannot be written as the tensor product $M = M_1 \bar{\otimes} M_2$ of two II_1 factors. Also, a maximal abelian von Neumann subalgebra $A \subset M$ is a *Cartan subalgebra* if its normalizer, $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) | uAu^* = A\}$, generates a weakly dense subalgebra of M .

The general goal of this paper is to provide new classes of II_1 factors that are prime and have at most one Cartan subalgebra. We start by giving a short history of results of this type.

In [Po83], Popa proved that uncountable free groups give rise to II_1 factors that are prime and do not have Cartan subalgebras. The first examples of separable such II_1 factors were obtained in the mid 90s as an application of free probability theory. Thus, Voiculescu showed that any II_1 factor admitting a generating set whose free entropy dimension is greater than 1 has no Cartan subalgebra [Vo95]. In particular, the free group factors, $L(\mathbb{F}_n)$, with $2 \leq n \leq \infty$, do not have Cartan subalgebras. Subsequently, Ge proved that the free group factors are also prime [Ge96].

During the last decade, Popa’s deformation/rigidity theory has generated spectacular progress in the study of II_1 factors. In particular, it has led to the first classes of II_1 factors that have a unique Cartan subalgebra, up to unitary conjugacy. We highlight here three major advances in this direction and refer the reader to the surveys [Po07, Va10, Io12b] for more information. In [Po01] Popa showed that any II_1 factor has at most one Cartan subalgebra which satisfies a certain combination of deformation and rigidity properties. Later on, Ozawa and Popa found the first class of II_1 factors that have a unique arbitrary Cartan subalgebra [OP07]. More precisely, they showed that any II_1 factor $L^\infty(X) \rtimes \mathbb{F}_n$ arising from a free ergodic profinite pmp action $\mathbb{F}_n \curvearrowright (X, \mu)$ of a free group \mathbb{F}_n ($2 \leq n \leq \infty$) has a unique Cartan subalgebra, up to unitary conjugacy. Very recently, Popa and Vaes vastly generalized this result by proving that it holds for any free ergodic pmp action $\mathbb{F}_n \curvearrowright (X, \mu)$ [PV11].

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In the last ten years, the primeness and absence of Cartan subalgebras of the free group factors [Vo95, Ge96] have been generalized and strengthened in many ways. Firstly, Ozawa proved that II_1 factors arising from hyperbolic groups Γ are *solid*: the relative commutant $A' \cap L(\Gamma)$ of any diffuse subalgebra $A \subset L(\Gamma)$ is amenable [Oz03]. In particular, $L(\Gamma)$ and all of its non-amenable subfactors are prime. Secondly, using a technique based on closable derivations, Peterson was able to show that II_1 factors arising from groups with positive first ℓ^2 -Betti number are prime [Pe06].

Using his deformation/rigidity theory, Popa then found a new proof of solidity for $L(\mathbb{F}_n)$ [Po06b]. Popa's approach relies on the remarkable discovery [Po06a] that the presence of spectral gap can be viewed as a source of rigidity. This *spectral gap rigidity* principle has since been the catalyst behind many developments in deformation/rigidity theory. Thus, it was a crucial ingredient in the finding of II_1 factors with a unique Cartan subalgebra [OP07, PV11]. In [OP07], Ozawa and Popa used the spectral gap rigidity principle to show that the free group factors enjoy a structural property, called *strong solidity*, which strengthens both solidity and absence of Cartan subalgebras: the normalizer of any diffuse subalgebra $A \subset L(\mathbb{F}_n)$ is amenable. Also using Popa's spectral gap rigidity principle, Chifan and Sinclair showed that, more generally, the group von Neumann algebra of any icc hyperbolic group is strongly solid [CS11].

1.2. Statement of main results. In this paper, we use Popa's deformation/rigidity theory to prove primeness and absence/uniqueness of Cartan subalgebras for II_1 factors in the presence of unbounded closable derivations satisfying certain regularity properties.

Our first result shows that if a non-amenable II_1 factor M admits a closable unbounded derivation into its coarse bimodule which has a “large” domain, then M is prime. To make this precise, let us introduce a definition. If M is a II_1 factor, then we say that a finite set $S \subset M$ is a *non-amenability set* if there exists a constant $K > 0$ such that $\|\xi\|_2 \leq K \sum_{x \in S} \|x\xi - \xi x\|_2$, for every vector $\xi \in L^2(M) \bar{\otimes} L^2(M)$. Note that by Connes' theorem [Co76], M is non-amenable if and only if it admits a non-amenability set.

Theorem 1.1. *Let M be a non-amenable II_1 factor and M_0 be a weakly dense $*$ -subalgebra which contains a non-amenability set for M . Assume that there exists a real closable unbounded derivation $\delta : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$.*

Then M is not L^2 -rigid. In particular, M is prime and does not have property Gamma.

Recall that if \mathcal{H} is an M - M bimodule, then a map $\delta : M_0 \rightarrow \mathcal{H}$ is a *derivation* if it verifies that $\delta(xy) = x\delta(y) + \delta(x)y$, for all $x, y \in M_0$. We say that δ is *bounded* if $\sup_{x \in M_0, \|x\| \leq 1} \|\delta(x)\| < \infty$. Note that δ is bounded if and only if it is *inner*, i.e. there exists $\xi \in \mathcal{H}$ such that $\delta(x) = x\xi - \xi x$, for all $x \in M_0$ (see the proof of [Pe04, Theorem 2.2]). Also, recall that a II_1 factor M has *property Gamma* of Murray and von Neumann [MvN43] if there exists a sequence $u_n \in \mathcal{U}(M)$ such that $\tau(u_n) = 0$, for all n , and $\|u_n x - x u_n\|_2 \rightarrow 0$, for all $x \in M$.

A II_1 factor M is *L^2 -rigid* in the sense of Peterson [Pe06] if any semigroup $\phi_t = \exp(-t\delta^*\bar{\delta})$ arising from a real closable densely defined derivation δ into a multiple of the coarse bimodule converges uniformly to id_M on the unit ball of M , as $t \rightarrow 0$. By [Pe06] if a II_1 factor is not prime or has property Gamma, then it is L^2 -rigid. On the other hand, if an icc group Γ admits an unbounded cocycle $b : \Gamma \rightarrow \ell^2(\Gamma)^{\oplus \infty}$, then $L(\Gamma)$ is not L^2 -rigid [Pe06]. Theorem 1.1 generalizes this fact and provides new examples of non- L^2 -rigid factors.

Note that if M_0 does not contain a non-amenability set for M , then Theorem 1.1 fails in general (see Remark 4.2).

By [Vo95, Ge96], II_1 factors which admit a set of generators whose microstates free entropy is finite, $\chi(X_1, \dots, X_n) > -\infty$, do not have property Gamma, are prime and do not have Cartan subalgebras. In [Vo98] Voiculescu introduced a non-microstates free entropy $\chi^*(X_1, \dots, X_n)$. The two entropies satisfy $\chi \leq \chi^*$ by [BCG03] and are believed to be equal, whenever Connes' embedding conjecture holds. Nevertheless, unlike its microstates counterpart, the non-microstates free entropy has not yet found applications to von Neumann algebras. In particular, it is an open problem whether the above indecomposability results hold under the assumption $\chi^*(X_1, \dots, X_n) > -\infty$. In this direction, it was shown in [Da08] that if the assumption that $\chi^*(X_1, \dots, X_n) > -\infty$ is replaced with the stronger assumption that the free Fisher information is finite $\Phi^*(X_1, \dots, X_n) < \infty$, then the von Neumann algebra M generated by $\{X_1, \dots, X_n\}$ is a II_1 factor without property Gamma.

In this paper, we show that if we further strengthen the condition $\Phi^*(X_1, \dots, X_n) < \infty$ then we can conclude that M is prime and does not have a Cartan subalgebra. Firstly, as a consequence of Theorem 1.1 we deduce the following.

Corollary 1.2. *Let (M, τ) be a tracial von Neumann algebra which is generated by $n \geq 2$ algebraically free self-adjoint elements X_1, \dots, X_n . Assume that either*

- $\mathcal{J}_p(X_i : \mathbb{C}\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \rangle)$ exists and belongs to M , for all $p \in \{1, 2\}$ and every $i \in \{1, \dots, n\}$, or
- $\Phi^*(X_1, \dots, X_n) < \infty$ and $n \geq 3$.

Then M is a non- L^2 -rigid II_1 factor. In particular, M is prime and does not have property Gamma.

To recall the definition of the p -th order conjugate variable $\xi_{p,i} = \mathcal{J}_p(X_i : \mathbb{C}\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \rangle)$, for $p \in \{1, 2\}$, let M_0 be the $*$ -algebra generated by X_1, \dots, X_n . Let $\delta_i : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ be the partial free difference quotient derivation given by $\delta_i(X_j) = \delta_{i,j} 1 \otimes 1$. Then the first and second order conjugate variables are defined as $\xi_{1,i} = \delta_i^*(1 \otimes 1) \in L^2(M)$ and $\xi_{2,i} = \delta_i^*(\xi_{1,i} \otimes 1) \in L^2(M)$, whenever these formulas make sense (see [Vo98, Definition 3.1]). If the first order conjugate variables $\xi_{1,i}$ exist, then the free Fisher information is given by $\Phi^*(X_1, \dots, X_n) = \sum_{i=1}^n \|\xi_{1,i}\|_2^2$.

Corollary 1.2 implies that if X_1, \dots, X_n are self-adjoint elements of a tracial von Neumann algebra (M, τ) , for some $n \geq 2$, and $S_1, \dots, S_n \in M$ are free semicircular elements which are free from X_1, \dots, X_n , then $X_1^\varepsilon = X_1 + \varepsilon S_1, \dots, X_n^\varepsilon = X_n + \varepsilon S_n$ generate a prime II_1 factor, for any $\varepsilon > 0$. Indeed, by [Vo98, Corollary 3.9] $X_1^\varepsilon, \dots, X_n^\varepsilon$ satisfy the above assumption on conjugate variables. Moreover, by using [Io12a], we can show that the II_1 factor generated by $X_1^\varepsilon, \dots, X_n^\varepsilon$ does not have a Cartan subalgebra (see Theorem 6.1). Note that in the case the von Neumann algebra generated by $\{X_1, \dots, X_n\}$ is embeddable into R^ω , the last two facts follow from [Vo95, Vo97, Ge96]. See Section 6 for indecomposability results for more general “regularized” algebras.

Secondly, by assuming a Lipschitz condition on conjugate variables [Da10b, Definition 1] we are able to deduce absence of Cartan subalgebras.

Theorem 1.3. *Let (M, τ) be a tracial von Neumann algebra which is generated by $n \geq 2$ algebraically free self-adjoint elements X_1, \dots, X_n . Let M_0 be the $*$ -algebra generated by X_1, \dots, X_n . For $1 \leq i \leq n$, denote by $\delta_i : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ the free difference quotient $\delta_i(X_j) = \delta_{i,j} 1 \otimes 1$.*

Let $\delta = (\delta_1, \dots, \delta_n) : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus n}$ and $\bar{\delta}$ be the closure of δ . Assume that $1 \otimes 1$ is in the domain of δ_i^ and denote $\xi_i = \delta_i^*(1 \otimes 1)$. Moreover, assume that ξ_i is in the domain of $\bar{\delta}$ and $\bar{\delta}(\xi_i) \in (M \bar{\otimes} M^{op})^{\oplus n}$, for all $1 \leq i \leq n$. Here, M^{op} denotes the opposite algebra of M , and we consider the inclusion $M \bar{\otimes} M^{op} \subset L^2(M \bar{\otimes} M^{op}) \cong L^2(M) \bar{\otimes} L^2(M)$.*

Then M is a II_1 factor which does not have a Cartan subalgebra. Moreover, $M \bar{\otimes} Q$ does not have a Cartan subalgebra, for any II_1 factor Q .

Theorem 1.3 generalizes a result of [Da10b] where by using microstates free entropy techniques and [Vo95, Sh07] it was shown that if M is embeddable into R^ω then it has no Cartan subalgebras.

In the second part of this paper we establish absence or uniqueness of Cartan subalgebras for II_1 factors M admitting certain unbounded “algebraic” derivations.

Firstly, we show that, under fairly general conditions, the existence of a finitely generated weakly dense $*$ -subalgebra $M_0 \subset M$, a von Neumann subalgebra $B \subset M$, and an unbounded derivation $\delta : M_0 \rightarrow L^2(\langle M, e_B \rangle)$ such that $\delta(M_0) \subset \text{span}(M_0 e_B M_0)$ implies that M has no Cartan subalgebras (see Theorem 7.1). Here, $\langle M, e_B \rangle$ denotes Jones’ basic construction.

Let us state two corollaries of this result. By Theorem 1.1 if a II_1 factor M admits a closable unbounded derivation $\delta : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ whose domain contains a non-amenability set, then it is prime. We believe that the existence of such a derivation δ should also imply that M does not have a Cartan subalgebra. However, proving this seems out of reach with the methods that are currently available. Nevertheless, as a consequence of Theorem 7.1 we are able to confirm this conjecture if δ is algebraic.

Corollary 1.4. *Let M be a non-amenable II_1 factor and $M_0 \subset M$ be a finitely generated weakly dense $*$ -subalgebra which contains a non-amenability set for M . Assume that there exists a real unbounded derivation $\delta : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ such that $\delta(M_0) \subset M_0 \otimes M_0$, $1 \otimes 1$ is in the domain of δ^* and $\delta^*(1 \otimes 1) \in M_0$.*

Then M does not have a Cartan subalgebra.

Corollary 1.4 generalizes part of [Io12a, Corollary 1.5]. Indeed, it implies that the free product $M = M_1 * M_2$ of any two finitely generated II_1 factors does not have a Cartan subalgebra. However, we are unaware of any example of a II_1 factor which satisfies the hypothesis of Corollary 1.4 and is not essentially a free product (see also Remark 3.5). Note that if M is embeddable into R^ω , then Corollary 1.4 also follows from [Sh07].

As a consequence of Theorem 7.1 we also provide a general criterion for absence of Cartan subalgebras in amalgamated free product II_1 factors.

Corollary 1.5. *Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with a common von Neumann subalgebra such that $\tau_1|_B = \tau_2|_B$ and denote $M = M_1 *_B M_2$. Assume that there exist unitary elements $u \in M_1$ and $v, w \in M_2$ such that $E_B(u) = E_B(v) = E_B(w) = E_B(w^*v) = 0$. Suppose that either $uBu^* \perp B$ or $vBv^* \perp B$.*

Then M is a II_1 factor, does not have a Cartan subalgebra and does not have property Gamma.

For the definition of the amalgamated free product of tracial von Neumann algebras, see [Po93] and [VDN92]. Following [Po83], we say that two von Neumann subalgebras B_1, B_2 of a tracial von Neumann algebra (M, τ) are orthogonal if $\tau(b_1 b_2) = \tau(b_1)\tau(b_2)$, for all $b_1 \in B_1$ and $b_2 \in B_2$.

Finally, we provide a new class of II_1 factors that have a unique Cartan subalgebra, up to unitary conjugacy. By [PV11, Definition 1.4] a countable group Γ , whose every free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ gives rise to a II_1 factor with a unique Cartan subalgebra, is called \mathcal{C} -rigid (Cartan rigid). In a recent breakthrough, Popa and Vaes proved that all weakly amenable groups with a positive first ℓ^2 -Betti number and all non-elementary hyperbolic groups are \mathcal{C} -rigid [PV11, PV12]. Most recently, it was shown in [Io12a, Theorem 1.1] that any amalgamated free product group $\Gamma = \Gamma_1 *_\Lambda \Gamma_2$ such that $[\Gamma_1 : \Lambda] \geq 2$, $[\Gamma_2 : \Lambda] \geq 3$, and $\cap_{i=1}^m g_i \Lambda g_i^{-1}$ is finite, for some $g_1, \dots, g_m \in \Gamma$, is \mathcal{C} -rigid.

Our next result gives a cohomological criterion, in terms of the existence of a certain unbounded algebraic cocycle, for a group Γ to be \mathcal{C} -rigid.

Theorem 1.6. *Let Γ be a countable icc group, $\Lambda < \Gamma$ be a subgroup and assume that there exists an unbounded cocycle $b : \Gamma \rightarrow \mathbb{C}(\Gamma/\Lambda)$ satisfying $b|_\Lambda \equiv 0$. Additionally, assume that*

- $\cap_{i=1}^m g_i \Lambda g_i^{-1}$ is finite, for some $g_1, g_2, \dots, g_m \in \Gamma$,
- there exists an increasing sequence $\{\Gamma_n\}_{n \geq 1}$ of finitely generated subgroups of Γ such that $\cup_{n \geq 1} \Gamma_n = \Gamma$ and $b(\Gamma_n) \subset \mathbb{C}(\Gamma_n \Lambda / \Lambda)$, for all $n \geq 1$,
- $L(\Gamma)$ does not have property Gamma and Λ is not co-amenable in Γ .

Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy, for any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$.

Since amalgamated free product groups admit such algebraic cocycles, this theorem generalizes [Io12a, Theorem 1.1] recalled above. Moreover, it leads to new examples of \mathcal{C} -rigid groups.

Corollary 1.7. *Let G be a countable group, $\Lambda < G$ be a subgroup, and $\theta : \Lambda \rightarrow G$ be an injective group homomorphism such that $\Lambda \neq G$ and $\theta(\Lambda) \neq G$. Denote by $\Gamma = \text{HNN}(G, \Lambda, \theta)$ the corresponding HNN extension. Assume that $\cap_{i=1}^m g_i \Lambda g_i^{-1}$ is finite, for some $g_1, g_2, \dots, g_m \in \Gamma$.*

Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy, for any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$.

Corollary 1.7 strengthens and generalizes the main result of [FV10]. Indeed, [FV10, Theorem 1.1] shows that if we moreover assume that G contains a non-amenable subgroup with the relative property (T) or two commuting non-amenable subgroups, and that Λ is amenable, then $L^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan subalgebra, for any free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$.

1.3. Comments on the proofs. Let us say a few words about the proofs of Theorems 1.1, Theorem 1.3 and Corollary 1.4, since they are representative of the proofs of all the results stated above. Consider a real unbounded closable derivation $\delta : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus n}$, for some $1 \leq n \leq \infty$, which satisfies the corresponding regularity conditions. Recall that M_0 contains a non-amenable set S for M . Our goal is to prove either that M is not L^2 -rigid (as in Theorem 1.1) or that M has no Cartan subalgebra (as in Theorem 1.3 and Corollary 1.4).

To this end, we use results from free probability theory to construct a malleable deformation of M , in the sense of Popa. This consists of a tracial von Neumann algebra \tilde{M} containing M and a pointwise $\|\cdot\|_2$ -continuous path $\{\alpha_t\}_{t \geq 0}$ of $*$ -homomorphisms $\alpha_t : M \rightarrow \tilde{M}$ such that $\alpha_0 = \text{id}_M$. Moreover, the pair $(\tilde{M}, \{\alpha_t\}_{t \geq 0})$ is a “dilation of δ ” in the following broad sense: the limit $\lim_{t \rightarrow 0} \frac{1}{t} \|\alpha_t(x) - x\|_2$ exists and is determined by δ , for all $x \in M_0$.

By [Da10a], any real closable derivation admits a dilation. Moreover, for derivations into a multiple of the coarse bimodule, such as δ , [Da10a] provides additional information on the dilation. More precisely, if M_t denotes the von Neumann algebra generated by M and $\alpha_t(M)$, then the M - M bimodule $L^2(M_t) \ominus L^2(M)$ is contained in a multiple of the coarse bimodule, for any $t \geq 0$.

If M is L^2 -rigid, then the semigroup $\phi_t = \exp(-t\delta^*\bar{\delta})$ converges uniformly to id_M on the unit ball of M , as $t \rightarrow 0$. This readily entails that α_t converges uniformly to id_M , as $t \rightarrow 0$. As a consequence, for every small enough t , there exists a unitary $u_t \in \tilde{M}$ such that $u_t \alpha_t(M) u_t^* \subset M$. Since $S \subset M$ is a non-amenable set, a variation of Popa’s spectral gap argument [Po06a] (see Lemma 2.7) allows us to find $K > 0$ such that

$$(*) \quad \|\alpha_t(x) - E_M(\alpha_t(x))\|_2 \leq K \sum_{y \in S} \|\alpha_t(y) - y\|_2, \text{ for all } x \in (M)_1 \text{ and } t \geq 0.$$

Since $S \subset M_0$ and α_t dilates δ , this inequality implies that δ is bounded. This is a contradiction, proving that M is not L^2 -rigid, as claimed by Theorem 1.1.

Now, assume that δ is the free difference quotient and the conjugate variables satisfy a Lipschitz condition (as in Theorem 1.3), or δ is algebraic (as in Corollary 1.4). Then results from [Da10b] and [Sh07] imply that the dilation algebra \tilde{M} can be taken equal to $M * L(\mathbb{F}_\infty)$. In this case, we say that δ admits a “free dilation”.

We then use techniques from Popa’s deformation/rigidity theory to prove that M does not have a Cartan subalgebra. In particular, we employ the recent work [Io12a] (which notably uses [PV11]) on the structure of normalizers of subalgebras of amalgamated free product algebras. By combining [Io12a] with Lemma 2.7 we show that if M has a Cartan subalgebra, then $(*)$ holds. As above, this provides a contradiction.

1.4. Organization of the paper. Besides the introduction, this paper has seven other sections. In Section 2 we record several notions and results that we will later use. In Section 3 we recall known results on dilating derivations. Sections 4-8 are devoted to the proofs of our main results.

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2. PRELIMINARIES

2.1. Terminology. We work with *tracial* von Neumann algebras (M, τ) , i.e. von Neumann algebras M endowed with a faithful, normal, tracial state τ . We denote by $\|x\|_2 = \tau(x^*x)^{1/2}$ the 2-norm associated to τ and by $\|x\|$ the operator norm. We denote by $\mathcal{Z}(M)$ the *center* of M , by $\mathcal{U}(M)$ the *group of unitaries* of M and by $(M)_1 = \{x \in M \mid \|x\| \leq 1\}$ the *unit ball* of M . We always assume that M is *separable*, unless it is a subalgebra of an ultraproduct algebra.

A tracial von Neumann algebra (M, τ) is called *amenable* if there exists a net $\xi_n \in L^2(M) \bar{\otimes} L^2(M)$ such that $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$ and $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$, for every $x \in M$. By Connes’ celebrated theorem [Co76], M is amenable if and only if it is approximately finite dimensional.

For a free ultrafilter ω on \mathbb{N} , the *ultraproduct* algebra M^ω is defined as the quotient $\ell^\infty(\mathbb{N}, M)/\mathcal{I}$, where $\mathcal{I} \subset \ell^\infty(\mathbb{N}, M)$ is the closed ideal of $x = (x_n)_n$ such that $\lim_{n \rightarrow \omega} \|x_n\|_2 = 0$. As it turns out, M^ω is a tracial von Neumann algebra, with its canonical trace given by $\tau_\omega((x_n)_n) = \lim_{n \rightarrow \omega} \tau(x_n)$.

If M and N are tracial von Neumann algebras, then an M - N *bimodule* is a Hilbert space \mathcal{H} endowed with commuting normal $*$ -homomorphisms $\pi : M \rightarrow \mathbb{B}(\mathcal{H})$ and $\rho : N^{op} \rightarrow \mathbb{B}(\mathcal{H})$. For $x \in M, y \in N$ and $\xi \in \mathcal{H}$ we denote $x\xi y = \pi(x)\rho(y)(\xi)$. If M, N, P are tracial von Neumann algebras, \mathcal{H} and \mathcal{K} be M - N and N - P bimodules, respectively, then $\mathcal{H} \otimes_N \mathcal{K}$ denotes the *Connes tensor product* endowed with the natural M - P bimodule structure (see [Po86]).

Let $Q \subset M$ be a von Neumann subalgebra. *Jones’ basic construction* $\langle M, e_Q \rangle$ is defined as the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q)$. Recall that $\langle M, e_Q \rangle$ has a faithful semi-finite trace given by $Tr(xe_Q y) = \tau(xy)$ for all $x, y \in M$. We denote by $L^2(\langle M, e_Q \rangle)$ the associated Hilbert space and endow it with the natural M -bimodule structure. Note that $L^2(\langle M, e_Q \rangle) \cong L^2(M) \otimes_Q L^2(M)$, as M - M bimodules.

Finally, if S is a subset of a von Neumann algebra \mathcal{M} , then a state ϕ on \mathcal{M} is said to be *S-central* if it satisfies $\phi(xT) = \phi(Tx)$, for all $x \in S$ and $T \in \mathcal{M}$.

2.2. Intertwining-by-bimodules. We next recall from [Po03, Theorem 2.1 and Corollary 2.3] Popa's powerful *intertwining-by-bimodules* technique (see also [Va06, Appendix C]).

Theorem 2.1. [Po03] *Let (M, τ) be a separable tracial von Neumann algebra and $P, Q \subset M$ be two (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

- *There exist non-zero projections $p \in P, q \in Q$, a $*$ -homomorphism $\phi : pPp \rightarrow qQq$ and a non-zero partial isometry $v \in qMp$ such that $\phi(x)v = vx$, for all $x \in pPp$.*
- *There is no sequence $u_n \in \mathcal{U}(P)$ satisfying $\|E_Q(xu_ny)\|_2 \rightarrow 0$, for all $x, y \in M$.*

If one of these conditions holds true, then we say that a corner of P embeds into Q inside M and write $P \prec_M Q$.

If M is not separable, then this statement holds true after we replace the sequence u_n with a net.

2.3. Relative amenability.

Definition 2.2. [OP07, Definition 2.2] Let (M, τ) be a tracial von Neumann algebra and let $P \subset pMp, Q \subset M$ be von Neumann subalgebras. We say that P is *amenable relative to Q inside M* if there exists a net $\xi_n \in L^2(p\langle M, e_Q \rangle p)$ such that $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$, for every $x \in pMp$, and $\|y\xi_n - \xi_ny\|_2 \rightarrow 0$, for every $y \in P$. By [OP07, Theorem 2.1], this condition is equivalent to the existence of a P -central state ϕ on $p\langle M, e_Q \rangle p$ such that $\phi|_{pMp} = \tau|_{pMp}$.

Remark 2.3. Let $\Lambda < \Gamma$ be countable subgroups. By [AD95, Proposition 3.5], $L(\Gamma)$ is amenable relative to $L(\Lambda)$ if and only if Λ is *co-amenable* in Γ : there is a Γ -invariant state on $\ell^\infty(\Gamma/\Lambda)$.

The failure of an algebra to be amenable (or amenable relative to some other algebra) can therefore be viewed as a source of “spectral gap rigidity”. The notion of spectral gap rigidity has been introduced by Popa and has been used to great effect for instance in [Po06a, Po06b, OP07]. Motivated by this, we introduce the following:

Definition 2.4. Let (M, τ) be a tracial von Neumann algebra and $P, Q \subset M$ be von Neumann subalgebras.

- (1) A finite set $S \subset M$ is called a *non-amenability set for M* if there exists a constant $K > 0$ such that $\|\xi\|_2 \leq K \sum_{y \in S} \|y\xi - \xi y\|_2$, for every $\xi \in L^2(M) \bar{\otimes} L^2(M)$.
- (2) A finite set $S \subset P$ is called a *non-amenability set for P relative to Q inside M* if there exists a constant $K > 0$ such that $\|\xi\|_2 \leq K \sum_{y \in S} \|y\xi - \xi y\|_2$, for every $\xi \in L^2(\langle M, e_Q \rangle)$.

Remark 2.5. If M has a non-amenability set, then M has no amenable direct summand. If M is a II_1 factor, then by Connes' theorem [Co76] the converse is true: M has a non-amenability set if and only if it is non-amenable. Note also that if there is a non-amenability set for P relative to Q , then Pp is not amenable relative to Q , for any projection $p \in P' \cap M$.

We will later need the following result which is an easy consequence of [OP07, Section 2.2] (see also [PV11, Section 2.5]).

Lemma 2.6. [OP07] *Let (M, τ) be a tracial von Neumann algebra and let $P, Q \subset M$ be von Neumann subalgebras. Let \mathcal{H} be a Q - P bimodule.*

- (1) *Assume that Pp is not amenable relative to Q , for any non-zero projection $p \in P' \cap M$. Then for any net of vectors $\xi_n \in L^2(M) \otimes_Q \mathcal{H}$ satisfying $\sup_n \|x\xi_n\| \leq \|x\|_2$, for all $x \in M$, and $\|y\xi_n - \xi_ny\| \rightarrow 0$, for all $y \in P$, we have that $\|\xi_n\| \rightarrow 0$.*
- (2) *If $S \subset P$ is a non-amenability set for P relative to Q inside M , then there exists a constant $\kappa > 0$ such that $\|\xi\| \leq \kappa \sum_{y \in S} \|y\xi - \xi y\|$, for all $\xi \in L^2(M) \otimes_Q \mathcal{H}$.*

Proof. The first assertion is a rephrasing of [Io12a, Lemma 2.3].

To prove the second assertion, let S be a non-amenability set for P relative to Q . Assuming that the conclusion fails, we can find a sequence of unit vectors $\xi_n \in L^2(M) \otimes_Q \mathcal{H}$ such that $\|y\xi_n - \xi_n y\| \rightarrow 0$, for all $y \in S$. Choose a state on $\ell^\infty(\mathbb{N})$, denoted \lim_n , extending the usual limit. Also, consider the normal $*$ -homomorphism $\pi : \langle M, e_Q \rangle \rightarrow \mathbb{B}(L^2(M) \otimes_Q \mathcal{H})$ given by $\pi(T)(\xi \otimes_Q \eta) = T(\xi) \otimes_Q \eta$.

Define $\psi : \langle M, e_Q \rangle \rightarrow \mathbb{C}$ by letting $\psi(T) = \lim_n \langle \pi(T)\xi_n, \xi_n \rangle$. Then ψ is a S -central state. Moreover, ψ is $C^*(S)$ -central, where $C^*(S)$ denotes the C^* -algebra generated by S . A standard procedure (see the proof of [OP07, Theorem 2.1]) implies the existence of a net of unit vectors $\eta_i \in L^2(\langle M, e_Q \rangle)$ such that $\|y\eta_i - \eta_i y\| \rightarrow 0$, for all $y \in \mathcal{U}(C^*(S))$. Thus, $\|y\eta_i - \eta_i y\| \rightarrow 0$, for all $y \in S$, contradicting the non-amenability of S . \square

The next lemma is a variant of Popa's spectral gap argument [Po06a]. It will be later used (e.g. in the proof of Theorem 1.1) to deduce boundedness of a derivation δ from the uniform convergence of the semigroup $\phi_t = \exp(-t\delta^*\delta)$.

Lemma 2.7. *Let (\tilde{M}, τ) be a tracial von Neumann algebra and M be a von Neumann subalgebra. Let $P, Q \subset M$ be von Neumann subalgebras. Assume that the M - M bimodule $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic to $L^2(M) \otimes_Q \mathcal{K}$, for some Q - M bimodule \mathcal{K} .*

Let $\alpha_n : M \rightarrow \tilde{M}$, $n \geq 1$, be trace preserving $$ -homomorphisms such that $\|\alpha_n(x) - x\|_2 \rightarrow 0$, for all $x \in M$. Assume that $p_n \in \alpha_n(P)' \cap \tilde{M}$ is a projection and $v_n \in \tilde{M}$ is a unitary such that $\alpha_n(P)p_n \subset v_n M v_n^*$, for all $n \geq 1$.*

- (1) *If Pp is not amenable relative to Q inside M , for every non-zero projection $p \in P' \cap M$, then $\sup_{x \in (P)_1} \|(\alpha_n(x) - E_M(\alpha_n(x)))p_n\|_2 \rightarrow 0$, as $n \rightarrow \infty$.*
- (2) *If $S \subset P$ is a non-amenability set for P relative to Q inside M , then there exists a constant $C > 0$ such that for all $n \geq 1$ we have*

$$\|(\alpha_n(x) - E_M(\alpha_n(x)))p_n\|_2 \leq C \sum_{y \in S} \|(\alpha_n(y) - y)p_n\|_2, \text{ for all } x \in (P)_1.$$

Proof. For $x \in P$ and $n \geq 1$, define $\beta_n(x) = v_n^* \alpha_n(x) p_n v_n \in M$. Denote $\mathcal{H} = L^2(\tilde{M}) \ominus L^2(M)$. Let \mathcal{H}_n be the M - P bimodule which is equal to \mathcal{H} endowed with the bimodule structure given by $y \cdot \xi \cdot x = y \xi \beta_n(x)$. Then \mathcal{H}_n is isomorphic to $L^2(M) \otimes_Q \mathcal{K}_n$, where \mathcal{K}_n is equal to \mathcal{K} endowed with the Q - P bimodule structure given by $y \cdot \xi \cdot x = y \xi \beta_n(x)$.

Define the Q - P bimodule $\tilde{\mathcal{K}} = \oplus_{n \geq 1} \mathcal{K}_n$ and let us treat separately the two assertions.

(1) In this case, Lemma 2.6 (1) implies that any net $\xi_n \in L^2(M) \otimes_Q \tilde{\mathcal{K}}$ satisfying $\|x \cdot \xi_n\| \leq \|x\|_2$ for every $x \in M$, and $\|x \cdot \xi_n - \xi_n \cdot x\| \rightarrow 0$, for all $x \in P$, must verify $\|\xi_n\| \rightarrow 0$. Define $\xi_n = p_n v_n - E_M(p_n v_n)$, then $\|x \xi_n\|_2 \leq \|x\|_2$, for all $x \in M$. If we view ξ_n as an element of \mathcal{H}_n , then for every $x \in P$ we have that

$$\begin{aligned} \|x \cdot \xi_n - \xi_n \cdot x\| &= \|x \xi_n - \xi_n \beta_n(x)\|_2 \leq \|x p_n v_n - p_n v_n \beta_n(x)\|_2 = \\ &= \|(x - \alpha_n(x))p_n v_n\|_2 \rightarrow 0. \end{aligned}$$

Since \mathcal{H}_n is isomorphic to a M - P sub-bimodule of $L^2(M) \otimes_Q \tilde{\mathcal{K}}$, we conclude that $\|\xi_n\|_2 \rightarrow 0$. It follows that for all $x \in (P)_1$ and $n \geq 1$ we have

$$\begin{aligned} \|\alpha_n(x)p_n - E_M(\alpha_n(x))p_n\|_2 &= \|\alpha_n(x)p_n v_n - E_M(\alpha_n(x))p_n v_n\|_2 \leq \\ 2\|\xi_n\|_2 + \|(1 - E_M)(\alpha_n(x)p_n v_n)\|_2 &= 2\|\xi_n\|_2 + \|(1 - E_M)(p_n v_n \beta_n(x))\|_2 \leq \end{aligned}$$

$$2\|\xi_n\|_2 + \|(1 - E_M)(p_nv_n)\|_2 = 3\|\xi_n\|_2.$$

Since $\|\xi_n\|_2 \rightarrow 0$ and $x \in (P)_1$ is arbitrary, this proves the first assertion.

(2) Assume that S is a non-amenability set for P relative to B . Lemma 2.6 (2) implies that we can find $\kappa > 0$ such that any vector $\xi \in L^2(M) \otimes_B \tilde{\mathcal{K}}$ verifies $\|\xi\| \leq \kappa \sum_{y \in S} \|y \cdot \xi - \xi \cdot y\|$. Thus, for all $n \geq 1$ and $\xi \in L^2(\tilde{M}) \ominus L^2(M)$, we have that $\|\xi\|_2 \leq \kappa \sum_{y \in S} \|y\xi - \xi\beta_n(y)\|_2$.

Denote $\delta_n = \sum_{y \in S} \|(\alpha_n(y) - y)p_n\|_2$. Then we have $\sum_{y \in S} \|yp_nv_n - p_nv_n\beta_n(y)\|_2 = \delta_n$ and thus

$$\sum_{y \in S} \|y(p_nv_n - E_M(p_nv_n)) - (p_nv_n - E_M(p_nv_n))\beta_n(y)\|_2 \leq \delta_n.$$

By combining the last two inequalities we conclude that $\|p_nv_n - E_M(p_nv_n)\|_2 \leq \kappa\delta_n$, for all $n \geq 1$. Together with the estimate from the proof of part (1), we get that if $x \in (P)_1$ and $n \geq 1$, then $\|(\alpha_n - E_M(\alpha_n(x)))p_n\|_2 \leq 3\|p_nv_n - E_M(p_nv_n)\|_2 \leq 3\kappa\delta_n$. Thus, the second assertion holds for $C = 3\kappa$. \square

2.4. Property Gamma. A II_1 factor M has *property Gamma* of Murray and von Neumann [MvN43] if there exists a sequence of unitaries $u_n \in M$ with $\tau(u_n) = 0$ such that $\|u_n x - x u_n\|_2 \rightarrow 0$, for all $x \in M$. If ω is a free ultrafilter on \mathbb{N} , then property Gamma is equivalent to $M' \cap M^\omega = \mathbb{C}1$. By a well-known result of Connes [Co76, Theorem 2.1] property Gamma is also equivalent to the existence of a net of unit vectors $\xi_n \in L^2(M) \ominus \mathbb{C}1$ satisfying $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$, for all $x \in M$.

Therefore, the failure of property Gamma implies the existence of a non-Gamma set in the sense of the following definition that was introduced in [Pe04, Definiton 3.1] and was also motivated by [Po86, Remark 4.1.6].

Definition 2.8. [Pe04] Let M be a II_1 factor. A finite set $S \subset M$ is called a *non-Gamma set* for M if there exists $K > 0$ such that $\|\xi\|_2 \leq \sum_{y \in S} K\|y\xi - \xi y\|_2$, for all $\xi \in L^2(M) \ominus \mathbb{C}1$.

Remark 2.9. By [Co76, Theorem 2.1] any II_1 factor M without property Gamma has a non-Gamma set. Note, however, that it is not always possible to find a non-Gamma set for M inside a given weakly dense $*$ -subalgebra of M . Recall that a countable group Γ is *inner amenable* if the unitary representation of Γ on $\ell^2(\Gamma \setminus \{e\})$ given by conjugation has almost invariant vectors. Vaes recently found an example of an icc group Γ which is inner amenable (hence $\mathbb{C}\Gamma$ does not contain a non-Gamma set for $L(\Gamma)$) such that $L(\Gamma)$ does not have property Gamma [Va09].

The next result follows easily from [Co76] but for the reader's convenience we include a proof.

Lemma 2.10. [Co76] Let M be a II_1 factor and $S \subset M$ be a finite set closed under adjoint.

If S is a non-Gamma set for M , then S is non-amenability set for M .

Proof. Let S be a non-Gamma set for M . Assume by contradiction that S is not a non-amenability set. Thus we can find a sequence of unit vectors $\xi_n \in L^2(M) \bar{\otimes} L^2(M)$ such that $\|x\xi_n - \xi_n x\|_2 \rightarrow 0$, for all $x \in S$. Choose a state on $\ell^\infty(\mathbb{N})$, denoted \lim_n , extending the usual limit. Define $\psi : \mathbb{B}(L^2(M)) \rightarrow \mathbb{C}$ by letting $\psi(T) = \lim_n \langle (T \otimes 1)\xi_n, \xi_n \rangle$.

Then ψ is an S -central state, hence $\phi = \psi|_M : M \rightarrow \mathbb{C}$ is an S -central state. Moreover, ϕ is central under the C^* -algebra $C^*(S)$ generated by S . Let $\eta_i \in L^1(M)$ be a net of positive norm one elements such that $\tau(x\eta_i) \rightarrow \phi(x)$, for all $x \in M$. Since ϕ is $C^*(S)$ -central, for all $u \in \mathcal{U}(C^*(S))$ we have that $\tau(x(u\eta_i u^* - \eta_i)) \rightarrow 0$, for all $x \in M$. Thus, $u\eta_i u^* - \eta_i \rightarrow 0$, in the weak topology, for all $u \in \mathcal{U}(C^*(S))$. The Hahn-Banach theorem implies that, after passing to convex combinations, we may assume that we have $\|u\eta_i u^* - \eta_i\|_1 \rightarrow 0$ in addition to $\tau(x\eta_i) \rightarrow \phi(x)$, for every $x \in M$.

The Powers-Størmer inequality (see [BO08, Proposition 6.2.4]) gives that $\|u\eta_i^{1/2}u^* - \eta_i^{1/2}\|_2 \rightarrow 0$, for all $u \in \mathcal{U}(C^*(S))$. Hence $\|y\eta_i^{1/2} - \eta_i^{1/2}y\|_2 \rightarrow 0$, for all $y \in S$. Since S is a non-Gamma set, we derive that $\|\eta_i^{1/2} - c_i \cdot 1\|_2 \rightarrow 0$, where $c_i = \langle \eta_i^{1/2}, 1 \rangle$. Applying Powers-Størmer again yields that $\|\eta_i - c_i^2 \cdot 1\|_1 \rightarrow 0$. This implies that $c_i^2 \tau(x) \rightarrow \phi(x)$, for all $x \in M$, hence $\phi = \tau$. Since ψ is $C^*(S)$ -central and $\psi|_M = \tau$, we get that ψ is central under the von Neumann algebra $W^*(S)$ generated by S . Thus $W^*(S)$ is amenable, contradicting the fact that it is a II_1 factor without property Gamma. \square

2.5. Mixing bimodules. Next, we recall the notion of mixing bimodules introduced in [PS09, Definition 2.3].

Definition 2.11. [PS09] Let (M, τ) be a tracial von Neumann algebra. We say that an M - M bimodule \mathcal{H} is *mixing* if for any sequence $a_n \in (M)_1$ such that $a_n \rightarrow 0$, weakly, we have

$$\sup_{x \in (M)_1} |\langle a_n \xi x, \eta \rangle| \rightarrow 0, \text{ and } \sup_{x \in (M)_1} |\langle x \xi a_n, \eta \rangle| \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ for all } \xi, \eta \in \mathcal{H}.$$

The coarse M - M bimodule $L^2(M) \bar{\otimes} L^2(M)$ is clearly mixing. Also, let Γ be a countable group and $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ be a mixing unitary representation. Then it is easy to see that $\mathcal{H} = \mathcal{K} \otimes \ell^2(\Gamma)$ is a mixing $L(\Gamma)$ - $L(\Gamma)$ bimodule (with its natural bimodule structure, defined as in [Po01, 1.1.4.]).

2.6. Normalizers of subalgebras of amalgamated free product von Neumann algebras. We will also need the following variant of [Io12a, Theorem 1.6] which is a hybrid between Theorems 1.6 and 5.2 from [Io12a].

Theorem 2.12. [Io12a] Let (M_1, τ_1) and (M_2, τ_2) be two tracial von Neumann algebras with a common von Neumann subalgebra B such that $\tau_1|_B = \tau_2|_B$ and denote $M = M_1 *_B M_2$. Let (Q, τ) be a tracial von Neumann algebra and $A \subset M \bar{\otimes} Q$ be an amenable von Neumann subalgebra. Denote by $P = \mathcal{N}_{M \bar{\otimes} Q}(A)''$ the von Neumann algebra generated by the normalizer of A in $M \bar{\otimes} Q$.

Assume that there are a group \mathcal{U} and homomorphisms $\rho_1 : \mathcal{U} \rightarrow \mathcal{U}(M), \rho_2 : \mathcal{U} \rightarrow \mathcal{U}(Q)$ such that

- $\rho_1(u) \otimes \rho_2(u) \in P$, for all $u \in \mathcal{U}$, and
- the von Neumann subalgebra $P_0 \subset M$ generated by $\rho_1(\mathcal{U})$ satisfies $P'_0 \cap M^\omega = \mathbb{C}1$.

Then one of the following conditions holds true:

- (1) $A \prec_{M \bar{\otimes} Q} B \bar{\otimes} Q$.
- (2) $P_0 \prec_M M_i$, for some $i \in \{1, 2\}$.
- (3) P_0 is amenable relative to B inside M .

Proof. For completeness, let us briefly indicate how the result follows from [Io12a].

Define $\mathcal{M} = M \bar{\otimes} Q$, $\mathcal{M}_1 = M_1 \bar{\otimes} Q$, $\mathcal{M}_2 = M_2 \bar{\otimes} Q$ and $\mathcal{B} = B \bar{\otimes} Q$. Then $\mathcal{M} = \mathcal{M}_1 *_B \mathcal{M}_2$. Further, we define $\tilde{\mathcal{M}} = \mathcal{M} *_B (\mathcal{B} \bar{\otimes} L(\mathbb{F}_2))$ and let $\{\theta_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\tilde{\mathcal{M}})$ be the free malleable deformation [IPP05] (see e.g. [Io12a, Section 2.5]). Let $\{u_g\}_{g \in \mathbb{F}_2}$ denote the canonical unitaries and define $N \subset \tilde{\mathcal{M}}$ to be the von Neumann subalgebra generated by $\cup_{g \in \mathbb{F}_2} u_g \mathcal{M} u_g^*$. Then N is normalized by $\{u_g\}_{g \in \mathbb{F}_2}$ and $\tilde{\mathcal{M}} = N \rtimes \mathbb{F}_2$.

Now, notice that if $t \in (0, 1)$ then $\theta_t(P) \subset \mathcal{N}_{\tilde{\mathcal{M}}}(\theta_t(A))$. S. Popa and S. Vaes' dichotomy [PV11, Theorem 1.6] implies that either $\theta_t(A) \prec_{\tilde{\mathcal{M}}} N$ or $\theta_t(P)$ is amenable relative to N . Thus, we conclude that we are in one of the following two cases:

Case 1. $\theta_t(A) \prec_{\tilde{\mathcal{M}}} N$, for some $t \in (0, 1)$.

Case 2. $\theta_t(P)$ is amenable relative to N , for all $t \in (0, 1)$.

In the first case, [Io12a, Theorem 3.2] implies that either $A \prec_{\mathcal{M}} \mathcal{B}$ or $P \prec_{\mathcal{M}} \mathcal{M}_i$, for some $i \in \{1, 2\}$. If the first alternative holds, then (1) is true. If $P \prec_{\mathcal{M}} \mathcal{M}_i$, then $P_0 \prec_M M_i$ and hence (2) is true. Indeed, if $P_0 \not\prec_M M_i$, then by the proof of [Po03, Corollary 2.3] we can find a sequence of unitaries $u_n \in \mathcal{U}$ such that $\|E_{M_i}(a\rho_1(u_n)b)\|_2 \rightarrow 0$, for all $a, b \in M$. But then it is clear that $\|E_{\mathcal{M}_i}(a(\rho_1(u_n) \otimes \rho_2(u_n))b)\|_2 \rightarrow 0$, for all $a, b \in \mathcal{M}$. This contradicts the assumption that $P \prec_{\mathcal{M}} \mathcal{M}_i$.

In the second case, [Io12a, Theorem 5.2] directly implies that either (2) or (3) hold. \square

3. DERIVATIONS AND FREE DILATIONS

In this section we record several results about derivations and their dilations.

Let (M, τ) be a tracial von Neumann algebra, $M_0 \subset M$ a weakly dense $*$ -subalgebra, and \mathcal{H} a M - M bimodule. A map $\delta : M_0 \rightarrow \mathcal{H}$ is a *derivation* if $\delta(xy) = x\delta(y) + \delta(x)y$, for all $x, y \in M_0$. We assume that δ is *closable* as an unbounded operator $\delta : L^2(M) \rightarrow \mathcal{H}$. We also suppose that δ is *real*, i.e. there exists a conjugate-linear isometric involution \mathcal{J} on \mathcal{H} satisfying $\mathcal{J}(x\delta(y)z) = z^*\delta(y^*)x^*$, for all $x, y, z \in M_0$. When $\mathcal{H} = L^2(\mathcal{M})$, for some semi-finite von Neumann algebra \mathcal{M} containing M , we assume that \mathcal{J} is given by $\mathcal{J}(x) = x^*$. In this case, δ is real if and only if $\delta(x^*) = \delta(x)^*$, for all $x \in M$.

Now, denote by $\bar{\delta}$ the closure of δ and by $D(\bar{\delta}) \subset L^2(M)$ its domain. By [Sa89] and [DL92], $D(\bar{\delta}) \cap M$ is a $*$ -subalgebra and $\bar{\delta}|_{D(\bar{\delta}) \cap M}$ is a derivation. Further, $\Delta = \delta^*\bar{\delta}$ gives rise to a semigroup of completely positive maps on M . More precisely, $\phi_t = \exp(-t\Delta) : M \rightarrow M$ are unital, trace preserving, completely positive maps satisfying $\phi_t \circ \phi_s = \phi_{t+s}$, for all $t, s > 0$, and $\|\phi_t(x) - x\|_2 \rightarrow 0$, as $t \rightarrow 0$, for every $x \in M$. Additionally, since δ is real, we have that ϕ_t is symmetric for every $t > 0$: $\tau(\phi_t(x)y) = \tau(x\phi_t(y))$, for all $x, y \in M$.

Recently it was proved that the semigroup $\{\phi_t\}_{t>0}$ admits a dilation in a larger tracial von Neumann algebra $\tilde{M} \supset M$ (see [Da10a, Theorem 24]). Here we state this result in the case when \mathcal{H} is a multiple of the coarse M - M bimodule. In this case, [Da10a, Proposition 26] provides additional information on certain M - M sub-bimodules of $L^2(\tilde{M})$.

Theorem 3.1. [Da10a] *Let (M, τ) be a tracial von Neumann algebra and $M_0 \subset M$ be a weakly dense $*$ -subalgebra. Let $\delta : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$ be a real closable derivation. Let $\Delta = \delta^*\bar{\delta}$ and consider the semigroup of completely positive maps $\phi_t = \exp(-t\Delta) : M \rightarrow M$.*

Then there exists a tracial von Neumann algebra \tilde{M} which contains M and $$ -homomorphisms $\alpha_t : M \rightarrow \tilde{M}$ such that $\phi_t = E_M \circ \alpha_t$, for all $t > 0$. Moreover, denote by $M_t \subset \tilde{M}$ the von Neumann subalgebra generated by M and $\alpha_t(M)$. Then the M - M bimodule $L^2(M_t) \ominus L^2(M)$ is isomorphic to a sub-bimodule of $(L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$, for every $t > 0$.*

In the sequel we will also need the following technical result.

Lemma 3.2. *Consider the notations from Theorem 3.1. Then for every $x \in D(\bar{\delta})$ we have that*

$$\frac{1}{t} \|\alpha_t(x) - \phi_t(x)\|_2^2 \rightarrow 2\|\delta(x)\|_2^2 \quad \text{and} \quad \frac{1}{t} \|\alpha_t(x) - x\|_2^2 \rightarrow 2\|\delta(x)\|_2^2, \quad \text{as } t \rightarrow 0.$$

Proof. Let $t > 0$ and recall $\phi_t = \exp(-t\Delta)$, where $\Delta = \delta^*\bar{\delta}$. By combining the identity $\text{id} - \phi_t = \int_0^t \Delta \circ \phi_s \, ds$ with the fact that $x \in D(\bar{\delta}) = D(\Delta^{1/2})$ we get that

$$\langle x - \phi_t(x), x \rangle = \int_0^t \langle \Delta(\phi_s(x)), x \rangle \, ds = \int_0^t \langle \phi_s(\Delta^{1/2}(x)), \Delta^{1/2}(x) \rangle \, ds.$$

Since $\phi_s(\Delta^{1/2}(x)) \rightarrow \Delta^{1/2}(x)$, in $\|\cdot\|_2$, as $s \rightarrow 0$, we conclude that

$$(3.1) \quad \frac{1}{t} \langle x - \phi_t(x), x \rangle \rightarrow \|\Delta^{1/2}(x)\|_2^2 = \|\delta(x)\|_2^2, \text{ as } t \rightarrow 0.$$

Finally, since $\phi_t(x) = E_M(\alpha_t(x))$, we get that $\|\alpha_t(x) - \phi_t(x)\|_2^2 = \|x\|_2^2 - \|\phi_t(x)\|_2^2 = \langle x - \phi_{2t}(x), x \rangle$. Also, we have that $\|\alpha_t(x) - x\|_2^2 = 2\langle x - \phi_t(x), x \rangle$. Together with equation 3.1 these identities yield the conclusion. \square

In the next section, the dilations from Theorem 3.1 will be used to prove that certain II_1 factors M are prime. On the other hand, in order to deduce that M does not have Cartan subalgebras, we will additionally need to know that the dilation “lives” in the free product $\tilde{M} = M * L(\mathbb{F}_\infty)$. In the rest of this section, we recall two results in this direction.

Shlyakhtenko showed that any “algebraic” derivation $\delta : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ gives rise, via exponentiation, to a one-parameter group of automorphisms of $M * L(\mathbb{Z})$ [Sh07, Proposition 2]. Here we note the following straightforward generalization of this result.

Proposition 3.3. *Let (M, τ) be a tracial von Neumann algebra, $B \subset M$ be a von Neumann subalgebra and $M_0 \subset M$ be a weakly dense $*$ -subalgebra. Assume that $\delta : D(\delta) \rightarrow L^2(\langle M, e_B \rangle)$ is a real derivation whose domain, denoted $D(\delta)$, contains both B and M_0 such that*

- $\delta(M_0) \subset \text{span}(M_0 e_B M_0)$.
- e_B is in the domain of δ^* and $\delta^*(e_B) \in M_0$.
- $\delta(b) = 0$, for all $b \in B$.

Assume that M_0 is finitely generated. More generally, assume that $M_0 = \cup_{n \geq 1} M_n$, where M_n are finitely generated $$ -algebras satisfying $M_n \subset M_{n+1}$ and $\delta(M_n) \subset \text{span}(M_n e_B M_n)$, for all $n \geq 1$. Denote $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$ and let $s \in L(\mathbb{Z})$ be a generating $(0, 1)$ semicircular element. Also, let $L^2(\langle M, e_B \rangle) \ni \xi \rightarrow \xi \# s \in \overline{\text{span}(M s M)}^{\|\cdot\|_2} \subset L^2(\tilde{M})$ be the unique isomorphism of M - M bimodules sending e_B to s .*

Then there exists a one-parameter group of automorphisms $\{\alpha_t\}_{t \in \mathbb{R}}$ of \tilde{M} such that

$$\left\| \frac{1}{t} (\alpha_t(x) - x) - \delta(x) \# s \right\|_2 \rightarrow 0, \text{ as } t \rightarrow 0, \text{ for all } x \in M_0.$$

The proof is an easy adaptation of the proof of [Sh07, Proposition 2] and can be derived by combining results from [Sh00, Section 3]. Nevertheless, for the reader’s convenience, we will sketch a proof.

Proof. If $b \in B$, then $\delta(b) = 0$. This implies that $\delta^*(e_B b) = \delta^*(e_B) b$ and $\delta^*(b e_B) = b \delta^*(e_B)$. Since $e_B b = b e_B$, we deduce that $[\delta^*(e_B), b] = 0$.

Let $D(\tilde{\delta})$ be the weakly dense $*$ -subalgebra of \tilde{M} generated by $M_0 \cup B \cup \{s\}$. Since $\delta(x) \# s \in D(\tilde{\delta})$, for all $x \in M_0 \cup B$, and $\delta^*(e_B) \in M_0$, we can define $\tilde{\delta} : M_0 \cup B \cup \{s\} \rightarrow D(\tilde{\delta})$ by letting

$$\tilde{\delta}(x) = \delta(x) \# s, \text{ for all } x \in M_0 \cup B, \text{ and } \tilde{\delta}(s) = -\delta^*(e_B).$$

Since $[\delta^*(e_B), B] = 0$ and $\delta|_B \equiv 0$, it is easy to see that $\tilde{\delta}$ extends to a derivation $\tilde{\delta} : D(\tilde{\delta}) \rightarrow D(\tilde{\delta})$. Also, since $\delta(x^*) = \delta(x)^*$, for all $x \in M_0$, and $\delta^*(e_B)$ is self-adjoint, we deduce that $\tilde{\delta}(x^*) = \tilde{\delta}(x)^*$, for all $x \in D(\tilde{\delta})$. Moreover, we have that:

Claim 1. $\tau(\tilde{\delta}(x)) = 0$, for all $x \in D(\tilde{\delta})$.

Proof of Claim 1. Denote by \mathcal{M} the $*$ -subalgebra of M generated by M_0 and B . For $n \geq 1$, define $s_n = s^n - \tau(s^n)$. Then $D(\tilde{\delta})$ is the linear span of

$$\mathcal{M} \cup \{x_1 s_{n_1} x_2 \dots x_k s_{n_k} x_{k+1} \mid x_1, x_{k+1} \in \mathcal{M}, x_2, \dots, x_k \in \mathcal{M} \ominus B, n_1, \dots, n_k \geq 1\}.$$

Since $\delta(M_0) \subset \text{span}(M_0 e_B M_0)$, we get that $\tilde{\delta}(\mathcal{M}) \subset \text{span}(\mathcal{M} s \mathcal{M})$. Hence, $\tau(\tilde{\delta}(x)) = 0$, for all $x \in \mathcal{M}$. Thus, in order to prove the claim, it suffices to show that $\tau(\tilde{\delta}(x)) = 0$, for every x of the form $x = x_1 s_{n_1} x_2 \dots x_k s_{n_k} x_{k+1}$, for some $k \geq 1$, $x_1, x_{k+1} \in \mathcal{M}, x_2, \dots, x_k \in \mathcal{M} \ominus B$ and $n_1, \dots, n_k \geq 1$.

Below, we sketch the proof of this fact in the case when k is even, leaving the (similar) case when k is odd to the reader. Assume therefore that $k = 2l$, for some $l \geq 1$.

Notice first that by freeness it follows that for all $i \in \{1, \dots, k+1\} \setminus \{l\}$, $y \in ML(\mathbb{Z})M$ and every $j \in \{1, \dots, k\} \setminus \{l, l+1\}$, $z \in L(\mathbb{Z})ML(\mathbb{Z})$, we have that

$$(3.2) \quad \tau(x_1 s_{n_1} \dots s_{n_i} y s_{n_{i+1}} \dots s_{n_k} x_{k+1}) = 0 \quad \text{and} \quad \tau(x_1 s_{n_1} \dots x_{n_j} z x_{n_{j+1}} \dots s_{n_k} x_{k+1}) = 0.$$

If $n \geq 1$, then $\tilde{\delta}(s_n) = \tilde{\delta}(s^n) = \sum_{i=0}^{n-1} s^i \tilde{\delta}(s) s^{n-1-i}$. Thus, $\tilde{\delta}(s_n) \in \text{span}(L(\mathbb{Z})ML(\mathbb{Z}))$. Also, recall that $\tilde{\delta}(x_0) \in \text{span}(ML(\mathbb{Z})M)$, for all $x_0 \in \mathcal{M}$. By combining these facts with equation 3.2, and using Leibniz's rule for $\tilde{\delta}$, it follows that

$$(3.3) \quad \tau(\tilde{\delta}(x)) = \tau(x_1 s_{n_1} \dots x_l \tilde{\delta}(s_{n_l} x_{l+1} s_{n_{l+1}}) x_{l+2} \dots s_{n_k} x_{k+1}).$$

Next, we denote by $\mathcal{K} \subset \tilde{M} \ominus B$ the set of alternating words in $M \ominus B$ and $L(\mathbb{Z}) \ominus \mathbb{C}1$, which start or begin with an element from $L(\mathbb{Z}) \ominus \mathbb{C}1$. Again, by freeness it is easy to see that

$$(3.4) \quad \tau(x_1 s_{n_1} \dots x_l y x_{l+2} \dots s_{n_k} x_{k+1}) = 0, \quad \text{for all } y \in \mathcal{K}.$$

Now, if $b \in B$, then $\tau(\tilde{\delta}(s)b) = \tau(\delta^*(e_B)b) = -\text{Tr}(e_B \delta(b)) = 0$ and thus $\tilde{\delta}(s) \in M \ominus B$. In combination with the formula for $\tilde{\delta}(s_n)$, we derive that $\tilde{\delta}(s_{n_l}), \tilde{\delta}(s_{n_{l+1}}) \in \text{span}(L(\mathbb{Z})(M \ominus B)L(\mathbb{Z}))$. Also, since $x_{l+1} \in \mathcal{M}$, we have that $\tilde{\delta}(x_{l+1}) \in \text{span}(M(L(\mathbb{Z}) \ominus \mathbb{C}1)M)$. Using these relations and Leibniz's rule it follows that $\tilde{\delta}(s_{n_l} x_{l+1} s_{n_{l+1}}) - \tau(\tilde{\delta}(s_{n_l} x_{l+1} s_{n_{l+1}}))$ belongs to the linear span of \mathcal{K} .

Combining this fact with 3.3 and 3.4 yields that $\tau(\tilde{\delta}(x)) = \tau(\tilde{\delta}(s_{n_l} x_{l+1} s_{n_{l+1}})) \tau(x_1 s_{n_1} \dots x_l x_{l+2} \dots s_{n_k} x_{k+1})$ and reduces Claim 1 to proving the following:

Claim 2. $\tau(\tilde{\delta}(s_m c s_n)) = 0$, for all $m, n \geq 1$ and $c \in \mathcal{M} \ominus B$.

Proof of Claim 2. First, since $\tilde{\delta}(s_n) = \sum_{i=0}^{n-1} s^i \tilde{\delta}(s) s^{n-1-i}$, for all $n \geq 1$, we get that

$$(3.5) \quad \tau(\tilde{\delta}(s_m c s_n)) = \sum_{i=0}^{m-1} \tau(s^i \tilde{\delta}(s) s^{m-1-i} c s_n) + \tau(s_m \tilde{\delta}(c) s_n) + \sum_{j=0}^{n-1} \tau(s_m c s^j \tilde{\delta}(s) s^{n-1-j}).$$

Note that $c, \tilde{\delta}(s) \in M \ominus B$ and $\tau(\tilde{\delta}(s)c) = -\tau(\delta^*(e_B)c)$. Since $\tilde{\delta}(c) \in \text{span}(MsM)$, for every $x \in L(\mathbb{Z})$ we have that

$$(3.6) \quad \begin{aligned} \tau(\tilde{\delta}(c)x) &= \tau(xs)\tau(\tilde{\delta}(c)s) = \tau(xs)\langle \delta(c) \# s, e_B \# s \rangle = \tau(xs)\langle \delta(c), e_B \rangle = \\ &= \tau(xs)\tau(c\delta^*(e_B)) = -\tau(\tilde{\delta}(s)c)\tau(xs). \end{aligned}$$

Altogether, by combining equations 3.5 and 3.6 we get that

$$(3.7) \quad \tau(\tilde{\delta}(s_m c s_n)) = \tau(\tilde{\delta}(s)c) \left[\sum_{i=0}^{m-1} \tau(s_n s^i) \tau(s^{m-1-i}) - \tau(s_n s_m s) + \sum_{j=0}^{n-1} \tau(s_m s^j) \tau(s^{n-1-j}) \right].$$

Now, let $\partial : \mathbb{C}\langle s \rangle \rightarrow \mathbb{C}\langle s \rangle \otimes \mathbb{C}\langle s \rangle$ be the difference quotient derivation given by $\partial(s) = 1 \otimes 1$. Since s is $(0, 1)$ semicircular, we have that $\partial^*(1 \otimes 1) = s$ (see [Vo98, Proposition 3.8]) and hence $(\tau \otimes \tau)(\partial(s^p)) = \tau(s^{p+1})$, for all p . Thus, we get that

$$\begin{aligned} & \sum_{i=0}^{m-1} \tau(s_n s^i) \tau(s^{m-1-i}) + \sum_{j=0}^{n-1} \tau(s_m s^j) \tau(s^{n-1-j}) = \\ & (\tau \otimes \tau)[s^n \partial(s^m) - \tau(s^n) \partial(s^m) + \partial(s^n) s^m - \tau(s^m) \partial(s^n)] = \\ & \tau(s^{m+n+1}) - \tau(s^n) \tau(s^{m+1}) - \tau(s^m) \tau(s^{n+1}). \end{aligned}$$

Since the last term is equal to $\tau(s_n s s_m)$ by equation 3.7 we conclude that $\tau(\tilde{\delta}(s_m c s_n)) = 0$. This finishes the proof of Claim 2 and hence of Claim 1. \square

Now, let $\tilde{M}_0 \subset \tilde{M}$ be the $*$ -subalgebra generated by $M_0 \cup \{s\}$. Then $\tilde{M}_0 \subset D(\tilde{\delta})$ and $\tilde{\delta}(\tilde{M}_0) \subset \tilde{M}_0$.

If M_0 is finitely generated, then \tilde{M}_0 is finitely generated. By using the fact that $\tilde{\delta}(x^*) = \tilde{\delta}(x)^*$, for all $x \in \tilde{M}_0$, and Claim 1, [Vo01, Proposition 3.3 and Corollary 3.7] imply that $\tilde{\delta}$ exponentiates to a one-parameter group $\alpha_t = e^{t\tilde{\delta}}$ of trace preserving automorphisms of \tilde{M} . Moreover, α_t satisfies the convergence required in the conclusion.

Finally, assume that M_0 is the increasing union of $*$ -algebras M_n satisfying $\delta(M_n) \subset \text{span}(M_n e_B M_n)$. Let $n_0 \geq 1$ such that $\delta^*(e_B) \in M_{n_0}$. For every n , let \tilde{M}_n be the $*$ -algebra generated by $M_n \cup \{s\}$. Then for every $n \geq n_0$, we have that $\tilde{\delta}(\tilde{M}_n) \subset \tilde{M}_n$. By the above, $\alpha_t = e^{t\tilde{\delta}}$ defines a one-parameter group of trace preserving automorphisms of the weak closure of \tilde{M}_n . Since the increasing union $\cup_{n \geq 1} \tilde{M}_n$ is weakly dense in \tilde{M} , the conclusion follows. \square

Remark 3.4. Proposition 3.3 can be used to recover several known constructions of malleable deformations, in the sense of Popa. Let us give two such examples (see also [Sh07, Example 1]).

(1) Let $M = M_1 *_B M_2$ be an amalgamated free product of tracial von Neumann algebras. Denote by $D(\delta)$ the $*$ -algebra generated by M_1 and M_2 . Define a derivation $\delta : D(\delta) \rightarrow L^2(\langle M, e_B \rangle)$ by letting $\delta(x) = i[x, e_B]$, if $x \in M_1$, and $\delta(x) = 0$, if $x \in M_2$. Then δ is real and satisfies the assumptions of Proposition 3.3. For $t \in \mathbb{R}$, let $u_t = \exp(its)$. The resulting one-parameter group of automorphisms of $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$ is given by $\alpha_t(x) = u_t x u_t^*$, if $x \in M_1$, and $\alpha_t(x) = x$, if $x \in M_2 \cup L(\mathbb{Z})$. This is a variant of the *free malleable deformation* of M introduced in [IPP05].

(2) Let $Q \subset P$ be tracial von Neumann algebras and $\theta : Q \rightarrow P$ be a $*$ -homomorphism. From this data, an HNN extension $M = \text{HNN}(P, Q, \theta)$ was constructed in [FV10, Section 3]. Briefly, M is a tracial von Neumann algebra generated by P and a unitary element u such that $uxu^* = \theta(x)$, for all $x \in Q$. Denote by $D(\delta)$ the $*$ -algebra generated by P and u . Then it is easy to see that $\delta : D(\delta) \rightarrow L^2(\langle M, e_Q \rangle)$ given by $\delta(x) = 0$, if $x \in P$, and $\delta(u) = iue_Q$, defines a real derivation which satisfies the assumptions of Proposition 3.3. For $t \in \mathbb{R}$, let $v_t = \exp(its)$. Then the one-parameter group of automorphisms of $\tilde{M} = M *_Q (Q \bar{\otimes} L(\mathbb{Z}))$ provided by Proposition 3.3 satisfies $\alpha_t(x) = x$, if $x \in P \cup L(\mathbb{Z})$, and $\alpha_t(u) = uv_t$. This recovers the malleable deformation of M introduced in [FV10, Section 3.5].

We are grateful to Jesse Peterson for pointing out to us the following remark.

Remark 3.5. In the case of group algebras, the existence of unbounded algebraic derivations implies strong restrictions on the structure of the group. Let Γ be an infinite, finitely generated countable group and assume that there exists an unbounded derivation $\delta : \mathbb{C}\Gamma \rightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$. Define $b : \Gamma \rightarrow \mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ by letting $b(g) = \delta(u_g)u_g^*$. Then we have that $b(gh) = b(g) + u_gb(h)u_g^*$, for all $g, h \in \Gamma$. Now, the representation of Γ on $\mathbb{C}\Gamma \otimes \mathbb{C}\Gamma$ by conjugation is isomorphic to the left regular representation λ of Γ on $\oplus_{n=1}^{\infty} \mathbb{C}\Gamma$. Thus, we obtain a cocycle $c = (c_n) : \Gamma \rightarrow \oplus_{n=1}^{\infty} \mathbb{C}\Gamma$.

Since δ is unbounded (hence not inner), it is easy to see that not all of the cocycles $c_n : \Gamma \rightarrow \mathbb{C}\Gamma$ can be inner. By [BV97, Lemma 2] this yields that Γ has at least two ends. Stallings' theorem now implies that Γ is either an amalgamated free product or an HNN extension over a finite subgroup. Thus, if Γ is moreover torsion free, then it is the free product $\Gamma = \Gamma_1 * \Gamma_2$ of two infinite groups.

We end this section with a result from [Da10b, Corollary 25] which shows that under a Lipschitz conjugate variables condition, the von Neumann algebra M generated by n self-adjoint elements X_1, \dots, X_n , admits a deformation into $M * L(\mathbb{F}_\infty)$.

Theorem 3.6. [Da10b] *Let (M, τ) be a tracial von Neumann algebra generated by $n \geq 2$ self-adjoint elements X_1, \dots, X_n . Let M_0 be the $*$ -algebra generated by X_1, \dots, X_n . For every $1 \leq i \leq n$, let $\delta_i : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ be the partial free difference quotient $\delta_i(X_j) = \delta_{i,j} X_i$. Denote $\delta = (\delta_1, \dots, \delta_n) : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus n}$ and let $\bar{\delta}$ be the closure of δ .*

Assume that $1 \otimes 1$ is in the domain of δ_i^ and denote by $\xi_i = \delta_i^*(1 \otimes 1)$ the corresponding conjugate variable. Moreover, assume that ξ_i is in the domain of $\bar{\delta}$ and $\bar{\delta}(\xi_i) \in (M \bar{\otimes} M^{\text{op}})^{\oplus n}$, for all $i \in \{1, \dots, n\}$. Here, M^{op} denotes the opposite algebra of M , and we consider the inclusion $M \bar{\otimes} M^{\text{op}} \subset L^2(M \bar{\otimes} M^{\text{op}}) \cong L^2(M) \bar{\otimes} L^2(M)$.*

Then for every $t \geq 0$, there exists a free family $S_1^{(t)}, \dots, S_n^{(t)} \in L(\mathbb{F}_\infty)$ of $(0, 1)$ -semicircular elements and a $$ -homomorphism $\alpha_t : M \rightarrow M * L(\mathbb{F}_\infty)$, such that*

$$\left\| \frac{1}{\sqrt{t}}(\alpha_t(x) - x) - \sum_{i=1}^n \delta_i(x) \# S_i^{(t)} \right\|_2 \rightarrow 0, \text{ as } t \rightarrow 0, \text{ for all } x \in M_0.$$

4. L^2 -RIGIDITY RESULTS

The main goal of this section is to prove Theorem 1.1. Let us first recall Peterson's notion of L^2 -rigidity for von Neumann algebras (see [Pe06, Definition 4.1 and Lemma 2.1]).

Definition 4.1. [Pe06] A tracial von Neumann algebra (M, τ) is L^2 -rigid if for any densely defined real closable derivation $\delta : D(\delta) \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$, the deformation $\phi_t = \exp(-t\delta^*\bar{\delta})$ converges uniformly to id_M on $(M)_1$, as $t \rightarrow 0$.

4.1. Proof of Theorem 1.1. By [Pe06, Corollary 4.6] any non-amenable II_1 factor that is non-prime or has property Gamma is L^2 -rigid. Thus, in order to get the conclusion, it suffices to prove that M is not L^2 -rigid. Assume by contradiction that M is L^2 -rigid.

Recall that $\delta : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ is a densely defined real derivation such that M_0 contains a non-amenable set S for M . For every $t > 0$, denote $\phi_t = \exp(-t\delta^*\bar{\delta})$. By Theorem 3.1 there exist a tracial von Neumann algebra \tilde{M} containing M and $*$ -homomorphisms $\alpha_t : M \rightarrow \tilde{M}$ such that $\phi_t = E_M \circ \alpha_t$, for all $t > 0$.

Since M is L^2 -rigid, ϕ_t converges uniformly to id_M on $(M)_1$. Thus, we can find $t_0 > 0$ such that $\|\phi_t(x) - x\|_2 \leq \frac{1}{2}$, for all $t \in [0, t_0]$ and every $x \in (M)_1$. Fix $t \in [0, t_0]$. Then for every $u \in \mathcal{U}(M)$ we have that $\tau(\alpha_t(u)u^*) = \tau(\phi_t(u)u^*) \geq \frac{1}{2}$. Denote by $K \subset (\tilde{M})_1$ the $\|\cdot\|_2$ -closure of the convex hull of the set $\{\alpha_t(u)u^* | u \in \mathcal{U}(M)\}$ and let $v_t \in K$ be the unique element of minimal $\|\cdot\|_2$. Then $\tau(v_t) \geq \frac{1}{2}$, hence $v_t \neq 0$, and $\alpha_t(u)v_t = v_t u$, for all $u \in \mathcal{U}(M)$.

Moreover, if we let M_t be the von Neumann algebra generated by $\alpha_t(M)$ and M , then $v_t \in M_t$ and $v_t^* v_t \in M' \cap M_t$. Thus, we get that $v_t^* v_t - E_M(v_t^* v_t) \in M' \cap M_t$. On the other hand, Theorem 3.1 gives that the M - M bimodule $L^2(M_t) \ominus L^2(M)$ is isomorphic to a sub-bimodule

of $(L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$. Since M is diffuse, we conclude that $v_t^* v_t \in M' \cap M = \mathbb{C}1$. Thus, by rescaling v_t , we get that there is a unitary $v_t \in M_t$ such that $\alpha_t(M) \subset v_t M v_t^*$, for every $t \in [0, t_0]$.

Let $t_n \in (0, t_0]$ be a sequence such that $t_n \rightarrow 0$. Since S is a non-amenability set for M , by Lemma 2.7 (2) we can find a constant $C > 0$ such that for all $n \geq 1$ we have

$$\|\alpha_{t_n}(x) - \phi_{t_n}(x)\|_2 = \|\alpha_{t_n}(x) - E_M(\alpha_{t_n}(x))\|_2 \leq C \sum_{y \in S} \|\alpha_{t_n}(y) - y\|_2, \text{ for all } x \in (M)_1.$$

Now, if we take $x \in M_0$, then Lemma 3.2 implies that $\frac{1}{\sqrt{t}} \|\alpha_t(x) - \phi_t(x)\|_2 \rightarrow \sqrt{2} \|\delta(x)\|_2$ and $\frac{1}{\sqrt{t}} \|\alpha_t(x) - x\|_2 \rightarrow \sqrt{2} \|\delta(x)\|_2$, as $t \rightarrow 0$. In combination with the last inequality this gives that $\|\delta(x)\|_2 \leq C \sum_{y \in S} \|\delta(y)\|_2$, for all $x \in M_0$. Thus, δ is bounded, which is a contradiction. \square

Remark 4.2. Let M be a II_1 factor and $p_n \in M$ a sequence of projections such that $\sum_{n=1}^{\infty} p_n = 1$. Then $M_0 = \cup_{n \geq 1} (p_1 + \dots + p_n)M(p_1 + \dots + p_n)$ is a weakly dense $*$ -subalgebra of M . Let α_n be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n^2 \tau(p_n)^2 = +\infty$. Then the map $\delta : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ given by $\delta(x) = \sum_{n=1}^{\infty} i\alpha_n [x, p_n \otimes p_n]$ is a well-defined derivation. Moreover, it is easy to see that δ is real, unbounded and closable. This shows that the assumption that M_0 contains a non-amenability set for M is necessary in the hypothesis of Theorem 1.1.

4.2. Proof of Corollary 1.2. We prove here Corollary 1.2 under the first assumption, and postpone dealing with the second assumption until Corollary 4.3. Denote by M_0 the $*$ -algebra generated by X_1, \dots, X_n . For every $i \in \{1, \dots, n\}$, let $\delta_i : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ be the partial free difference quotient derivation given by $\delta_i(X_j) = \delta_{i,j} 1 \otimes 1$. Further, let $\delta = (\delta_1, \dots, \delta_n) : M_0 \rightarrow (L^2(M) \bar{\otimes} L^2(M))^{\oplus n}$. Since $\delta_i^*(1 \otimes 1) = \mathcal{J}_1(X_i : \mathbb{C}\langle X_1, \dots, \hat{X}_i, \dots, X_n \rangle)$ exists and belongs to $L^2(M)$, [Vo98, Corollary 4.1] implies that δ_i is a closable derivation, for all $i \in \{1, 2, \dots, n\}$. Therefore, δ is real and closable.

By [Da08, Theorem 13], M is a II_1 factor without property Gamma. Moreover, since the first and second variables are bounded, [Da08, Lemmas 9 and 10] imply that $S = \{X_1, \dots, X_n\}$ is a non-Gamma set for M (see [Da08, Remark 11]). By Lemma 2.10 we therefore deduce that S is a non-amenability set for M .

Since $\Phi^*(X_1, \dots, X_n) = \sum_{i=1}^n \|\delta_i^*(1 \otimes 1)\|_2^2 < \infty$, the distribution of X_i has no atoms, for every i . Indeed, by [Vo98, Proposition 7.9], the free entropy χ^* satisfies $\chi^*(X_1, \dots, X_n) > -\infty$. Since $\chi^*(X_1) + \dots + \chi^*(X_n) \geq \chi^*(X_1, \dots, X_n)$ (see [Vo98, Proposition 7.3]), we deduce that $\chi^*(X_i) > -\infty$, for all i . Finally, since $\chi^*(X_i) = \chi(X_i)$ (see [Vo98, Proposition 7.6]), by [Vo94, Proposition 4.5] we get that the distribution of X_i has no atoms, for all $i \in \{1, \dots, n\}$.

As in [Pe04, Section 1.6] it follows that δ is not inner. Moreover, [Pe04, Theorem 2.2] implies that δ is unbounded. Altogether, we can apply Theorem 1.1 and deduce the conclusion. \square

Let (M, τ) be a tracial von Neumann algebra generated by $n \geq 2$ algebraically free $*$ -subalgebras A_1, \dots, A_n . The next result uses the liberation Fisher information $\varphi^*(A_1; \dots; A_n)$ introduced in [Vo99, Definition 9.1]. The definition involves the derivations $\delta_{A_i} : D(\delta_{A_i}) \rightarrow L^2(M) \bar{\otimes} L^2(M)$, where $D(\delta_{A_i})$ is the algebra generated by A_1, \dots, A_n , $\delta_{A_i}(a) = a \otimes 1 - 1 \otimes a$, if $a \in A_i$, and $\delta_{A_i}(a) = 0$, if $a \in A_j$ for $j \neq i$. See [Vo99, Section 5] for more details.

Assuming that $\varphi^*(A_1; \dots; A_n) = \sum_{i=1}^n \|\delta_{A_i}^*(1 \otimes 1)\|_2^2 < \infty$, Voiculescu proved that δ_{A_i} is a closable derivation [Vo99, Corollary 6.3]. Note that the combination of Remark 9.2 (f), Proposition 5.9 (a) and Definition 9.1 from [Vo99] gives that

$$(4.1) \quad \varphi^*(A_1; W^*(A_2, \dots, A_n)) = \varphi^*(A_1; \mathbb{C}\langle A_2, \dots, A_n \rangle) \leq 2\varphi^*(A_1; \dots; A_n)$$

Finally, we will also need the following relation between Fisher information and liberation Fisher information: if $\Phi^*(X_1, \dots, X_n) < \infty$ then $\varphi^*(W^*(X_1); \dots; W^*(X_n)) < \infty$ [Vo99, Corollary 5.11].

Corollary 4.3. *Let (M, τ) be a tracial von Neumann algebra which is generated by $n \geq 2$ non-trivial ($\neq \mathbb{C}1$) von Neumann subalgebras A_1, \dots, A_n . Assume that A_1, \dots, A_n have finite liberation Fisher information $\varphi^*(A_1; \dots; A_n) < \infty$, A_1 is diffuse, and A_2 is a non-amenable II_1 factor.*

Then M is a non L^2 -rigid II_1 factor. Thus, M is prime, does not have property Γ nor property (T) . In particular, this is the case if M is generated by $m \geq 3$ self-adjoint elements X_1, \dots, X_m satisfying $\Phi^(X_1, \dots, X_m) < \infty$.*

Proof. Since A_1 is diffuse, by arguing as in [Da08, Theorem 1] it follows that M is a factor. Indeed, let $x \in \mathcal{Z}(M)$ and define $\eta_\alpha : M \rightarrow M$ by $\eta_\alpha = \alpha(\alpha + \delta_{A_2}^* \delta_{A_2})^{-1}$, for $\alpha > 0$. If $y \in A_1$, then $\delta_{A_2}(y) = 0$, hence $\eta_\alpha(y) = y$. This implies that $\eta_\alpha(yx) = y\eta_\alpha(x)$ and $\eta_\alpha(xy) = \eta_\alpha(x)y$. In particular, we get that $[\delta_{A_2}(\eta_\alpha(x)), y] = \delta_{A_2}(\eta_\alpha([x, y])) = 0$, for all $\alpha > 0$ and $y \in A_1$. Since A_1 is diffuse and $\delta_{A_2}(\eta_\alpha(x))$ can be viewed as a Hilbert-Schmidt operator, we deduce that $\delta_{A_2}(\eta_\alpha(x)) = 0$. Since $\|\eta_\alpha(x) - x\|_2 \rightarrow 0$, as $\alpha \rightarrow \infty$, and δ_{A_2} is closable, we get that $x \in D(\bar{\delta}_{A_2})$ and $\bar{\delta}_{A_2}(x) = 0$. Thus, for every $y \in A_2$, we have $0 = \bar{\delta}_{A_2}([x, y]) = [x, y \otimes 1 - 1 \otimes y] = [y, [x, 1 \otimes 1]]$. Now, since A_2 is diffuse we get that $[x, 1 \otimes 1] = 0$, which implies that $x \in \mathbb{C}1$, as claimed.

Since A_2 is a non-amenable II_1 factor, by [Co76] we can find a non-amenable set $S \subset A_2$. But then S is a non-amenable set for M which is contained in $D(\delta_{A_2})$. Since A_1 is diffuse, δ_{A_2} is not inner and hence is not bounded by [Pe04, Theorem 2.2]. Thus, we can apply Theorem 1.1 to derive the conclusion.

Now, assume that M is generated by $m \geq 3$ self-adjoint elements with $\Phi^*(X_1, \dots, X_m) < \infty$. Let A_1 be the von Neumann algebra generated by X_1 and A_2 the von Neumann algebra generated by X_2, \dots, X_m . Since $\Phi^*(X_1, \dots, X_m) < \infty$, as in the proof of Corollary 1.2, we deduce that the distributions of X_1, \dots, X_m have no atoms. As a consequence, A_1 and A_2 are diffuse. Moreover, by [Da08, Theorem 13], A_2 is a non-amenable II_1 factor.

Since $\Phi^*(X_1, \dots, X_m) < \infty$, we have that $\varphi^*(W^*(X_1); \dots; W^*(X_n)) < \infty$. By using equation 6.3 we further get that $\varphi^*(A_1; A_2) \leq 2\varphi^*(W^*(X_1); \dots; W^*(X_m)) < \infty$. Altogether, the above shows that M is non L^2 -rigid factor. \square

5. LIPSCHITZ CONJUGATE VARIABLES AND ABSENCE OF CARTAN SUBALGEBRAS

This section is mainly devoted to the proof Theorem 1.3.

5.1. Proof of Theorem 1.3. We claim that the Lipschitz condition implies that the first and second conjugate variables are bounded. Firstly, Voiculescu (see equation (1) in [Da08]) noticed that for all $x \in M_0 = \mathbb{C}\langle X_1, \dots, X_n \rangle$ we have

$$(5.1) \quad \sum_{i=1}^n (\delta_i(x)(X_i \otimes 1) - (1 \otimes X_i)\delta_i(x)) = x \otimes 1 - 1 \otimes x.$$

Moreover, this identity holds for every $x \in D(\bar{\delta})$. Thus, if $x \in D(\bar{\delta})$ satisfies $\bar{\delta}(x) \in (M \bar{\otimes} M^{op})^{\oplus n}$, then $x = \tau(x) + E_{M \bar{\otimes} 1}(\sum_{i=1}^n (\delta_i(x)(X_i \otimes 1) - (1 \otimes X_i)\delta_i(x)))$ and therefore $x \in M$. This implies that $\mathcal{J}_1(X_i : \mathbb{C}\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \rangle) = \xi_i = \delta_i^*(1 \otimes 1) \in M$, for every $i \in \{1, \dots, n\}$. Now, recall that $\mathcal{J}_2(X_i : \mathbb{C}\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \rangle) = \delta_i^*(\xi_i \otimes 1)$. Since by the second formula in [Vo98, Proposition 4.1] (applied to $a = \xi_i$ and $\eta = 1 \otimes 1$) we have that $\delta_i^*(\xi_i \otimes 1) = \xi_i^2 - (\tau \otimes \text{id})(\bar{\delta}_i(\xi_i))$, we conclude that $\mathcal{J}_2(X_i : \mathbb{C}\langle X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \rangle) \in M$, for all $i \in \{1, \dots, n\}$. This proves our claim.

Since $\Phi^*(X_1, \dots, X_n) = \sum_{i=1}^n \|\delta_i^*(1 \otimes 1)\|_2^2 < \infty$, [Da08, Theorem 13] implies that M is a II_1 factor without property Gamma. Moreover, since the first and second order conjugate variables are bounded, $S = \{X_1, \dots, X_n\}$ is a non-Gamma set for M (see [Da08, Lemma 9 and 10, and Remark 11]). Lemma 2.10 gives that S is a non-amenability set for M .

Let Q be either $\mathbb{C}1$ or a II_1 factor. Our aim is to show that $M \bar{\otimes} Q$ does not have a Cartan subalgebra. Assume by contradiction that there is a Cartan subalgebra $A \subset M \bar{\otimes} Q$. Denote $\tilde{M} = M * L(\mathbb{F}_\infty)$. Let $\alpha_t : M \rightarrow \tilde{M}$ be the $*$ -homomorphisms provided by Theorem 3.6. We extend α_t to a $*$ -homomorphism $\alpha_t : M \bar{\otimes} Q \rightarrow \tilde{M} \bar{\otimes} Q$ by letting $\alpha_t|_Q = \text{id}_Q$.

The rest of the proof is divided between three claims.

Claim 1. Given $t \geq 0$, there exist projections $p_t, q_t, r_t \in \mathcal{Z}(\alpha_t(M)' \cap \tilde{M})$ satisfying $p_t + q_t + r_t = 1$,

- (1) $u_t \alpha_t(M) p_t u_t^* \subset M$, for some unitary $u_t \in \tilde{M}$,
- (2) $v_t \alpha_t(M) q_t v_t^* \subset L(\mathbb{F}_\infty)$, for some unitary $v_t \in \tilde{M}$,
- (3) $\alpha_t(M) r_t \not\prec_{\tilde{M}} M$ and $\alpha_t(M) r_t \not\prec_{\tilde{M}} L(\mathbb{F}_\infty)$.

Proof of Claim 1. Since M and $L(\mathbb{F}_\infty)$ are factors, the sets of projections satisfying (1) and (2) are closed taking supremum. Let $p_t, q_t \in \mathcal{Z}(\alpha_t(M)' \cap \tilde{M})$ be the maximal projections satisfying (1) and (2). [IPP05, Theorem 1.2.1] implies that $p_t q_t = 0$. Let $r_t = 1 - p_t - q_t$. If $\alpha_t(M) r_t \prec_{\tilde{M}} M$, then, as M is a factor, the proof of [IPP05, Theorem 5.1] yields a non-zero projection $r \in \mathcal{Z}(\alpha_t(M)' \cap \tilde{M}) r_t$ and a unitary $u \in \tilde{M}$ such that $u \alpha_t(M) r u^* \subset M$. Since $r \leq r_t$, this contradicts the maximality of p_t . Similarly, $\alpha_t(M) r_t \prec_{\tilde{M}} L(\mathbb{F}_\infty)$ contradicts the maximality of q_t . \square

Claim 2. $r_t = 0$, for all $t \geq 0$.

Proof of Claim 2. Suppose by contradiction that $r_t \neq 0$, for some $t \geq 0$.

Since M is a non-amenable II_1 factor, by using (3) from Claim 1 and applying [Io12a, Theorem 6.3] to the inclusion $\alpha_t(M) r_t \subset \tilde{M} = M * L(\mathbb{F}_\infty)$ we get that $(\alpha_t(M) r_t)' \cap (r_t \tilde{M} r_t)^\omega \prec_{\tilde{M}^\omega} \mathbb{C}1$. Thus, there exists a non-zero projection $r'_t \in \mathcal{Z}((\alpha_t(M) r_t)' \cap (r_t \tilde{M} r_t)^\omega) = \mathcal{Z}(\alpha_t(M)' \cap \tilde{M}^\omega) r_t$ such that $r'_t (\alpha_t(M)' \cap \tilde{M}^\omega) r'_t$ is completely atomic. By [Io12a, Lemma 2.7] it follows that $r'_t \in \tilde{M}$ and moreover $r'_t (\alpha_t(M)' \cap \tilde{M}^\omega) r'_t = r'_t (\alpha_t(M)' \cap \tilde{M}) r'_t$.

By combining these facts we deduce that there exists a non-zero projection $r''_t \in \alpha_t(M)' \cap \tilde{M}$ such that $r''_t \leq r'_t$ and $r''_t (\alpha_t(M)' \cap \tilde{M}^\omega) r''_t = \mathbb{C} r''_t$. Thus, we have $(\alpha_t(M) r''_t)' \cap (r''_t \tilde{M} r''_t)^\omega = \mathbb{C} r''_t$.

Next, we denote still by r''_t the projection $r''_t \otimes 1 \in \tilde{M} \bar{\otimes} Q$. We define $A_1 = \alpha_t(A) r''_t$ and note that $P_1 := \mathcal{N}_{r''_t(M \bar{\otimes} Q) r''_t}(A_1)''$ contains $(\alpha_t(M) \bar{\otimes} Q) r''_t$. In particular, P_1 contains $P_0 := \alpha_t(M) r''_t \otimes 1$. Since $(\alpha_t(M) r''_t)' \cap (r''_t \tilde{M} r''_t)^\omega = \mathbb{C} r''_t$, by applying Theorem 2.12 we get that one of the following conditions holds: (a) $A_1 \prec_{\tilde{M} \bar{\otimes} Q} 1 \otimes Q$, (b) $P_1 \prec_{\tilde{M}} M$ or $P_1 \prec_{\tilde{M}} L(\mathbb{F}_\infty)$, or (c) P_0 is amenable.

Now, it is easy to see that (a) implies that $A \prec_{M \bar{\otimes} Q} 1 \otimes Q$. By taking relative commutants (see e.g. [Va08, Lemma 3.5]) it follows that $M \otimes 1 \prec_{M \bar{\otimes} Q} A$. Since M is non-amenable while A is abelian, this is a contradiction. By using (3) from Claim 1 and the fact that M is non-amenable, we get that (b) and (c) cannot hold either. This altogether provides the desired contradiction. \square

Claim 3. $\tau(q_t) \rightarrow 0$, as $t \rightarrow 0$.

Proof of Claim 3. Since the M - M bimodule $L^2(\langle \tilde{M}, e_{L(\mathbb{F}_\infty)} \rangle)$ is isomorphic to $(L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$ and M is non-amenable, we get that M is not amenable relative to $L(\mathbb{F}_\infty)$ inside \tilde{M} . Since M is a factor, we also have that $M' \cap \tilde{M} = \mathbb{C}1$ (see [Po83, Remark 6.3]). By [OP07, Corollary 2.3] we derive that for any net of vectors $\xi_n \in L^2(\langle \tilde{M}, e_{L(\mathbb{F}_\infty)} \rangle)$ satisfying $\|x \xi_n\|_2 \leq \|x\|_2$, for all $x \in \tilde{M}$, and $\|y \xi_n - \xi_n y\|_2 \rightarrow 0$, for all $y \in M$, we must have that $\|\xi_n\|_2 \rightarrow 0$.

Let $\xi_t = v_t^* e_{L(\mathbb{F}_\infty)} v_t q_t \in L^2(\langle \tilde{M}, e_{L(\mathbb{F}_\infty)} \rangle)$, for any $t \geq 0$. Since $v_t \alpha_t(M) q_t v_t^* \subset L(\mathbb{F}_\infty)$, we get that $\alpha_t(y) \xi_t = \xi_t \alpha_t(y)$, for all $y \in M$. Also, $\|x \xi_t\|_2, \|\xi_t x\|_2 \leq \|x\|_2$, for all $x \in \tilde{M}$ and every $t \geq 0$. Thus, if $y \in M$, then $\|y \xi_t - \xi_t y\|_2 \leq 2\|y - \alpha_t(y)\|_2$. Since $\|\alpha_t(y) - y\|_2 \rightarrow 0$, by using the previous paragraph we conclude that $\|\xi_t\|_2 \rightarrow 0$, as $t \rightarrow 0$. Since $\|\xi_t\|_2^2 = \tau(q_t)$, we are done. \square

We are now ready to derive a contradiction. Recall that S is a non-amenability set for M and the M - M bimodule $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic to $(L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$. Since $u_t \alpha_t(M) p_t u_t^* \subset M$, Lemma 2.7 (2) implies that we can find $C > 0$ such that for all $t \geq 0$ we have

$$(5.2) \quad \|(\alpha_t(x) - E_M(\alpha_t(x)))p_t\|_2 \leq C \sum_{y \in S} \|(\alpha_t(y) - y)p_t\|_2, \quad \text{for all } x \in (M)_1.$$

Now, let M_0 be the $*$ -algebra generated by $\{X_1, \dots, X_n\}$ and fix $x \in M_0$ with $\|x\| \leq 1$. Theorem 3.6 gives that $\|\frac{1}{\sqrt{t}}(\alpha_t(x) - x) - \sum_{i=1}^n \delta_i(x) \# S_i^{(t)}\|_2 \rightarrow 0$. Note also that $E_M(\delta_i(x) \# S_i^{(t)}) = 0$ and $\|\sum_{i=1}^n \delta_i(x) \# S_i^{(t)}\|_2 = \sqrt{\sum_{i=1}^n \|\delta_i(x)\|_2^2} = \|\delta(x)\|_2$. Claims 2 and 3 together imply that $\|p_t - 1\|_2 \rightarrow 0$. By combining all of these facts and 5.2 we deduce that $\|\delta(x)\|_2 \leq C \sum_{y \in S} \|\delta(y)\|_2$. Since $x \in (M_0)_1$ is arbitrary, we deduce that δ is bounded. As in the proof of Corollary 1.2, this leads to a contradiction. \square

6. PRIMENESS AND ABSENCE OF CARTAN SUBALGEBRAS FOR REGULARIZED ALGEBRAS

In this section we establish indecomposability results for algebras obtained by free additive convolution and liberation. Firstly, for free additive convolution by semicircular variables $\{S_1, \dots, S_n\}$, we prove that algebras of the form $M_\varepsilon = \{X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n\}''$ are prime and do not have Cartan subalgebras. More precisely, we have

Theorem 6.1. *Let (M, τ) be a tracial von Neumann algebra and $X_1, \dots, X_n \in M$ be $n \geq 2$ self-adjoint elements. Let $\{S_1, \dots, S_n\} \in L(\mathbb{F}_n)$ be the canonical semicircular family and $\varepsilon > 0$. Denote by $M_\varepsilon \subset M * L(\mathbb{F}_n)$ the von Neumann subalgebra generated by $X_1 + \varepsilon S_1, \dots, X_n + \varepsilon S_n$.*

Then M_ε is a non- L^2 -rigid II_1 factor that does not have a Cartan subalgebra.

Proof. Fix $\varepsilon > 0$. Then by [Vo98, Corollary 3.9] we have that $\mathcal{J}_p(X_i^\varepsilon : \mathbb{C}\langle X_1^\varepsilon, \dots, X_{i-1}^\varepsilon, X_{i+1}^\varepsilon, \dots, X_n^\varepsilon \rangle)$ exists and belongs to M_ε , for all $p \in \{1, 2\}$ and $i \in \{1, \dots, n\}$. By Corollary 1.2 it follows that M_ε is a non- L^2 -rigid II_1 factor.

Now, assume by contradiction that M_ε has a Cartan subalgebra and denote $\tilde{M} = M * L(\mathbb{F}_n)$. After enlarging M if necessary (e.g. by replacing it with $M * L(\mathbb{F}_2)$) we may assume that it is a factor. Since M_ε is a non-amenable factor and $M, L(\mathbb{F}_n)$ are factors, by [Io12a, Corollary 9.1] we can find unitary elements $u, v \in \tilde{M}$ and projections $p, q \in \mathcal{Z}(M'_\varepsilon \cap \tilde{M})$ such that $u M_\varepsilon p u^* \subset M$, $v M_\varepsilon q v^* \subset L(\mathbb{F}_n)$ and $p + q = 1$. Since $L(\mathbb{F}_n)$ is strongly solid [OP07], we must have that $q = 0$, hence $u M_\varepsilon u^* \subset M$.

Towards a contradiction, let $i \in \{1, \dots, n\}$. We define A_i, M_i and N_i to be the von Neumann subalgebras of \tilde{M} generated by $\{X_i + \varepsilon S_i\}$, $M \cup \{S_i\}$ and $\{S_j\}_{j \in \{1, \dots, n\} \setminus \{i\}}$, respectively. Following [Vo93, Proposition 4.7], the distribution of $X_i + \varepsilon S_i$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} , hence it has no atoms. This implies that A_i is diffuse.

Since $u A_i u^* \subset M \subset M_i$, $A_i \subset M_i$ and $\tilde{M} = M_i * N_i$, by applying [Po83] or [IPP05, Theorem 1.2.1] we derive that $u \in M_i$. Since $i \in \{1, \dots, n\}$ was arbitrary, we conclude that $u \in \cap_{i=1}^n M_i$. Now, freeness easily yields that $\cap_{i=1}^n M_i = M$ and therefore $u \in M$. Thus, we would get that $M_\varepsilon \subset M$. This would imply that $S_i \in M$, for all $i \in \{1, \dots, n\}$, thus giving a contradiction. \square

Our techniques allow us to more generally handle the case of regularization by variables $\{Y_1, \dots, Y_n\}$ that have bounded first and second order conjugate variables.

Theorem 6.2. *Let $(M_1, \tau_1), (M_2, \tau_2)$ be tracial von Neumann algebras and let $M = M_1 * M_2$. Let $X_1 \in M_1 \setminus \mathbb{C}1, X_2, \dots, X_n \in M_1$ and $Y_1, \dots, Y_n \in M_2 \setminus \mathbb{C}1$ be self-adjoint elements, for some $n \geq 2$. Denote by $N \subset M$ the von Neumann subalgebra generated by $Z_1 = X_1 + Y_1, \dots, Z_n = X_n + Y_n$.*

Assume that Y_1, \dots, Y_n have bounded first and second order conjugate variables, i.e. we have that $\mathcal{J}_p(Y_i : \mathbb{C}\langle Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n \rangle)$ exists and belongs to M_2 , for all $p \in \{1, 2\}$ and $i \in \{1, \dots, n\}$.

Then N is a non- L^2 -rigid II_1 factor that does not have a Cartan subalgebra.

Proof. Without loss of generality, we may assume that M_1 and M_2 are factors, and that $\tau(Y_i) = 0$, for all $i \in \{1, \dots, n\}$. Let $M_0 \subset M$ be the $*$ -subalgebra generated by M_1 and $\{Y_1, \dots, Y_n\}$. We define two real closable derivations $\delta_1, \delta_2 : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$, by letting $\delta_1(x) = 0$ and $\delta_2(x) = i[x, 1 \otimes 1]$, if $x \in M_1$, and $\delta_1(y) = i[y, 1 \otimes 1]$ and $\delta_2(y) = 0$, if $y \in M_2$.

We continue by proving three facts.

Firstly, let $i \in \{1, \dots, n\}$. Since Y_1, \dots, Y_n have bounded first and second order conjugate variables, Y_i has finite free entropy. By [Vo94, Proposition 4.5], the distribution of Y_i is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} . In particular, the distribution of Y_i has no atoms.

Secondly, by [Vo98, Proposition 3.7] we have that the conjugate variables

$$\mathcal{J}_p(Z_i : \mathbb{C}\langle Z_1, \dots, Z_{i-1}, Z_{i+1}, \dots, Z_n \rangle) = E_N(\mathcal{J}_p(Y_i : \mathbb{C}\langle Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n \rangle))$$

exist and belong to N , for every $p \in \{1, 2\}, i \in \{1, \dots, n\}$. By combining [Da08, Remark 11] and Lemma 2.10 we get that $F = \{Z_1, \dots, Z_n\}$ is a non-Gamma, hence a non-amenability set for N .

Thirdly, let us prove that $\cap_{i=1}^n L^2(\{M_1, Y_i\}'') = L^2(M_1)$. We start by considering the free difference quotient $\partial_i : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ with respect to Y_i given by $\partial_i(x) = 0$, for $x \in M_1$, and $\partial_i(Y_j) = \delta_{i,j} 1 \otimes 1$, for $j \in \{1, \dots, n\}$. By [Vo98, Proposition 3.6], $\partial_i^*(1 \otimes 1)$ exists and is equal to $\mathcal{J}_1(Y_i : \mathbb{C}\langle Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n \rangle)$. By [Vo98, Corollary 4.2] we deduce that $M_0 \otimes M_0 \subset D(\partial_i^*)$.

Now, let $Z \in \cap_{i=1}^n L^2(\{M_1, Y_i\}'')$. Towards proving that $Z \in L^2(M_1)$, take a sequence $Z_n \in M_0$ such that $\|Z_n - Z\|_2 \rightarrow 0$. If $i \in \{1, \dots, n\}$, then $Z \in D(\bar{\partial}_i)$ and $\bar{\partial}_i(Z) = 0$. Since $M_0 \otimes M_0 \subset D(\partial_i^*)$ we deduce that $\langle \partial_i(Z_n), \zeta \rangle \rightarrow 0$, for all $\zeta \in M_0 \otimes M_0$. Also, a variant of an identity due to Voiculescu (see [Da08, equation (1)]) gives that $\delta_1(Z_n) = i \sum_{j=1}^n [\partial_j(Z_n)(X_j \otimes 1) - (1 \otimes X_j) \partial_j(Z_n)]$.

We therefore conclude that $\langle \delta_1(Z_n), \zeta \rangle \rightarrow 0$, for all $\zeta \in M_0 \otimes M_0$. This implies that $Z \in D(\bar{\delta}_1)$ and $\bar{\delta}_1(Z) = 0$. From this it is easy to see that $Z \in L^2(M_1)$.

Suppose that N either has a Cartan subalgebra or is L^2 -rigid. Towards getting a contradiction, we first use the fact that F is contained in the domains of δ_1 and δ_2 to prove the following:

Claim 1. There exists $\xi \in L^2(M) \bar{\otimes} L^2(M)$ such that $\delta_1(x) = [x, \xi]$, for all $x \in N \cap M_0$.

Proof of Claim 1. Let $\tilde{M} = M * L(\mathbb{Z})$ and $s \in L(\mathbb{Z})$ be a generating $(0, 1)$ semicircular element. For $t \in \mathbb{R}$ and $j \in \{1, 2\}$, we define $\alpha_t^{(j)} \in \text{Aut}(\tilde{M})$ by letting $\alpha_t^{(j)}(x) = x$, if $x \in M_j$, and $\alpha_t^{(j)}(y) = \exp(its)y \exp(-its)$, if $y \in M_k * L(\mathbb{Z})$, where k is such that $\{j, k\} = \{1, 2\}$. Then it is easy to see (e.g., by combining Theorem 3.3 and Remark 3.4) that

$$(6.1) \quad \left\| \frac{1}{t} (\alpha_t^{(j)}(x) - x) - \delta_j(x) \# s \right\|_2 \rightarrow 0, \quad \text{as } t \rightarrow 0, \quad \text{for all } x \in M_0.$$

To get the conclusion, we treat separately the two cases. Assume first that N has a Cartan subalgebra. Since N is a non-amenable factor and M_1, M_2 are factors, [Io12a, Corollary 9.1]

implies that we can find unitary elements $u_1, u_2 \in M$ and projections $p_1, p_2 \in \mathcal{Z}(N' \cap M)$ such that $u_1 N p_1 u_1^* \subset M_1$, $u_2 N p_2 u_2^* \subset M_2$ and $p_1 + p_2 = 1$.

Let $t \in \mathbb{R}$ and $i \in \{1, 2\}$. Then $\alpha_t^{(i)}(u_i) \alpha_t^{(i)}(N) \alpha_t^{(i)}(p_i) \alpha_t^{(i)}(u_i)^* \subset \alpha_t^{(i)}(M_i) \subset M$. Also, the M - M bimodule $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic to $(L^2(M) \bar{\otimes} L^2(M))^{\oplus \infty}$. Since F is a non-amenability set for N , Lemma 2.7 (2) provides a constant $C > 0$ such that for all $x \in (N)_1$ we have

$$(6.2) \quad \|(\alpha_t^{(i)}(x) - E_M(\alpha_t^{(i)}(x))) \alpha_t^{(i)}(p_i)\|_2 \leq C \sum_{y \in F} \|\alpha_t^{(i)}(y) - y\|_2.$$

Note that $E_M(\delta_i(x) \# s) = 0$, for all $x \in M_0$. Thus, combining equations 6.1 and 6.2 yields that $\|\delta_i(x) p_i\|_2 \leq C \sum_{y \in F} \|\delta(y)\|_2$, for all $x \in (N \cap M_0)_1$. Since $\delta_1(x) p_2 = i[x, 1 \otimes 1] p_2 - \delta_2(x) p_2$, we get that $\|\delta_1(x) p_2\|_2 \leq 2 + C \sum_{y \in F} \|\delta(y)\|_2$, for all $x \in (N \cap M_0)_1$. Since $p_1 + p_2 = 1$, it follows that the restriction of δ_1 to $N \cap M_0$ is bounded. The conclusion follows as in the proof of [Pe04, Theorem 2.2].

Now, assume that N is L^2 rigid. Note that the restriction of δ_1 to $N \cap M_0$ is a real closable derivation into $L^2(M) \bar{\otimes} L^2(M)$. Since $L^2(M) \bar{\otimes} L^2(M)$ is isomorphic to a N - N sub-bimodule of $(L^2(N) \bar{\otimes} L^2(N))^{\oplus \infty}$ and $N \cap M_0$ contains a non-amenability set for N , Theorem 1.1 implies that δ_1 is bounded on $N \cap M_0$. The conclusion now follows as above. \square

Now, by the definition of δ_1 we have that $\delta_1(Z_j) = i[Y_j, 1 \otimes 1]$. Denote $\eta = -i\xi \in L^2(M) \bar{\otimes} L^2(M)$. Since Claim 1 yields that $\delta_1(Z_i) = [Z_i, \xi]$, we conclude that

$$(6.3) \quad [Y_i, 1 \otimes 1] = [Z_i, \eta], \quad \text{for all } i \in \{1, \dots, n\}.$$

Next, we identify $L^2(M) \bar{\otimes} L^2(M)$ with $L^2(M \bar{\otimes} M^{op})$ in the natural way such that the M - M bimodule structure of $L^2(M) \bar{\otimes} L^2(M)$ corresponds to the left multiplication action of $M \bar{\otimes} M^{op}$ on $L^2(M \bar{\otimes} M^{op})$. We can therefore rewrite 6.3 as

$$(6.4) \quad Y_i \otimes 1 - 1 \otimes Y_i^{op} = (Z_i \otimes 1 - 1 \otimes (Z_i)^{op}) \eta, \quad \text{for all } i \in \{1, \dots, n\}.$$

Claim 2. $\eta \in L^2(M_1) \bar{\otimes} L^2(M_1) \cong L^2(M_1 \bar{\otimes} M_1^{op})$.

Proof of Claim 2. Fix $i \in \{1, \dots, n\}$ and denote by $M^{(i)} = \{M_1, Y_i\}''$ the von Neumann algebra generated by M_1 and Y_i . Since $X_i \in M_1 \setminus \mathbb{C}1$ and the distribution of Y_i has no atoms, employing [Be06, Theorem 4.1] we derive that the distribution of $Z_i = X_i + Y_i$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R} and thus has no atoms. This implies that the self-adjoint element $T_i := Z_i \otimes 1 - 1 \otimes (Z_i)^{op} \in M \bar{\otimes} M^{op}$ has no kernel.

For $\delta > 0$, let $A_\delta = \{r \in \mathbb{R} \mid |r| \geq \delta\}$ and $f_\delta : \mathbb{R} \rightarrow \mathbb{R}$ be the Borel function $f_\delta(r) = \frac{1}{r} 1_{A_\delta}(r)$. Since T_i has no kernel, $f_\delta(T_i) T_i = 1_{A_\delta}(T_i) \rightarrow \text{id}$, in the strong operator topology, as $\delta \rightarrow 0$. Thus, by using formula 6.4 we derive that $\|f_\delta(T_i)(Y_i \otimes 1 - 1 \otimes Y_i^{op}) - \eta\|_2 = \|(f_\delta(T_i) T_i - 1) \eta\|_2 \rightarrow 0$, as $\delta \rightarrow 0$. Since $Y_i \in M^{(i)}$ and $T_i \in M^{(i)} \bar{\otimes} M^{(i)op}$, we get that $f_\delta(T_i)(Y_i \otimes 1 - 1 \otimes Y_i^{op}) \in M^{(i)} \bar{\otimes} M^{(i)op}$.

We deduce that $\eta \in L^2(M^{(i)} \bar{\otimes} M^{(i)op}) \cong L^2(M^{(i)} \bar{\otimes} L^2(M^{(i)}))$, for all $i \in \{1, \dots, n\}$. Since we proved that $\cap_{i=1}^n L^2(M^{(i)}) = L^2(M_1)$, we conclude that $\eta \in L^2(M_1) \bar{\otimes} L^2(M_1)$, as claimed. \square

Let P be the orthogonal projection from $L^2(M) \bar{\otimes} L^2(M)$ onto $L^2(M) \bar{\otimes} L^2(M_1)$. Equation 6.3 gives that $Y_1 \otimes 1 - 1 \otimes Y_1 = Z_1 \eta - \eta Z_1$. Since $E_{M_1}(Y_1) = 0$ and $\eta \in L^2(M_1) \bar{\otimes} L^2(M_1)$, by applying P to the last identity, we deduce that $Y_1 \otimes 1 = Z_1 \eta - \eta X_1$. Hence $1 \otimes Y_1 = \eta Y_1$. Since Y_1 is diffuse, this implies that $\eta = 1 \otimes 1$ and further that $X_1 \otimes 1 = 1 \otimes X_1$. Thus, $X_1 \in \mathbb{C}1$, which is the desired contradiction. \square

Finally, we prove an indecomposability result for regularized algebras obtained by liberation in the sense of [Vo99, Section 2].

Theorem 6.3. *Let (M_1, τ_1) , (M_2, τ_2) be tracial von Neumann algebras and $M = M_1 * M_2$. Let $A_1, \dots, A_n \subset M_1$ be diffuse von Neumann subalgebras and $u_1, \dots, u_n \in M_2$ be unitary elements, for some $n \geq 2$. Denote by $N \subset M$ the von Neumann subalgebra generated by $u_1 A_1 u_1^*, \dots, u_n A_n u_n^*$.*

Assume that A_1 is a non-amenable II_1 factor and that $u_2 \notin \mathbb{C}u_1$.

Then N is a non- L^2 -rigid II_1 factor that does not have a Cartan subalgebra.

Proof. Let us first show that N is a factor. Let $x \in \mathcal{Z}(N)$. Then $[u_1^* x u_1, A_1] = [u_2^* x u_2, A_2] = 0$. Since $A_1, A_2 \subset M_1$ are diffuse, applying [Po83] or [IPP05, Theorem 1.2.1] gives that $u_1^* x u_1$ and $u_2^* x u_2$ belong to M_1 . Equivalently, $x \in u_1 M_1 u_1^* \cap u_2 M_1 u_2^*$. Since $u_1, u_2 \in M_2$ and $u_2 \notin \mathbb{C}u_1$, we get that $x \in \mathbb{C}1$.

Next, let $M_0 \subset M$ be the $*$ -subalgebra generated by M_1 and M_2 . Consider the real closable derivation $\delta_1 : M_0 \rightarrow L^2(M) \bar{\otimes} L^2(M)$ by letting $\delta_1(x) = 0$, if $x \in M_1$, and $\delta_1(y) = i[y, 1 \otimes 1]$, if $y \in M_2$. Since A_1 is a non-amenable II_1 factor, [Co76] implies that there exists a non-amenability set $F \subset A_1$. Thus, $u_1 F u_1^*$ is a non-amenability set for N which is contained in the domain of δ_1 .

If N is either L^2 -rigid or has a Cartan subalgebra, then the proof of Theorem 6.2 implies that there exists $\xi \in L^2(M) \bar{\otimes} L^2(M)$ such that $\delta_1(x) = [x, \xi]$, for all $x \in N \cap M_0$.

To get a contradiction, let $j \in \{1, \dots, n\}$. Since A_j is diffuse, we can find a self-adjoint element $X_j \in A_j$ which generates a diffuse algebra. If we define $Z_j = u_j X_j u_j^*$ then we have

$$[Z_j, \xi] = \delta_1(u_j X_j u_j^*) = i([u_j, 1 \otimes 1] X_j u_j^* + u_j X_j [u_j^*, 1 \otimes 1]) = i[Z_j, 1 \otimes 1 - u_j \otimes u_j^*].$$

Since Z_j is diffuse, we deduce that $\xi = i(1 \otimes 1 - u_j \otimes u_j^*)$ for all $j \in \{1, \dots, n\}$. This implies in particular that $u_1 \otimes u_1^* = u_2 \otimes u_2^*$, and hence that $u_2 \in \mathbb{C}u_1$, which gives a contradiction. \square

7. ALGEBRAIC DERIVATIONS AND ABSENCE OF CARTAN SUBALGEBRAS

The main goal of this section is to establish a general result showing absence of Cartan subalgebras for any II_1 factor M admitting certain unbounded “algebraic” derivations. More precisely:

Theorem 7.1. *Let M be a II_1 factor, $B \subset M$ be a von Neumann subalgebra and $M_0 \subset M$ be a weakly dense $*$ -subalgebra, such that M_0 contains a non-amenability set for M relative to B . Assume that for any non-zero projection $r \in B' \cap M$, there exists a mixing B - B sub-bimodule \mathcal{H} of $L^2(M)$ such that $r \mathcal{H} r \neq \{0\}$. Let $D(\delta) \subset M$ be a $*$ -subalgebra which contains M_0 and B .*

Assume that there exists a real derivation $\delta : D(\delta) \rightarrow L^2(\langle M, e_B \rangle)$ such that $\delta|_{M_0}$ is unbounded, $\delta^(e_B)$ exists and belongs to M_0 , and $\delta(b) = 0$, for all $b \in B$.*

Also, suppose that M_0 is finitely generated and $\delta(M_0) \subset \text{span } M_0 e_B M_0$. More generally, suppose that $M_0 = \cup_{n \geq 1} M_n$, where M_n is a finitely generated $$ -subalgebra such that $M_n \subset M_{n+1}$ and $\delta(M_n) \subset \text{span } M_n e_B M_n$, for all $n \geq 1$.*

Then M has no Cartan subalgebra and does not have property Gamma.

The mixingness condition was inspired by [Ho12, Corollary C], where it is shown that if an orthogonal representation $\pi : G \rightarrow \mathcal{O}(H_{\mathbb{R}})$ contains a mixing subrepresentation, then the II_1 factor $\Gamma(H_{\mathbb{R}})'' \rtimes G$ associated to the corresponding free Bogoljubov action has no Cartan subalgebra.

Before proceeding to the proof of Theorem 7.1, let us derive several consequences of it. Firstly, note that Corollary 1.4 corresponds precisely to the case $B = \mathbb{C}1$. Secondly, let us deduce Corollary 1.5.

7.1. Proof of Corollary 1.5. Recall that $M = M_1 *_B M_2$. Let $D(\delta)$ be the $*$ -algebra generated by M_1 and M_2 and define $\delta : D(\delta) \rightarrow L^2(\langle M, e_B \rangle)$ by letting $\delta(x) = i[x, e_B]$, if $x \in M_1$, and $\delta(x) = 0$, if $x \in M_2$. Then it is easy to see that δ is a real derivation and $\delta^*(e_B) = 0$.

Let $M_{1,n} \subset M_1$ and $M_{2,n} \subset M_2$ be increasing sequences of finitely generated $*$ -subalgebras such that $M_{1,0} = \cup_{n \geq 1} M_{1,n}$ is weakly dense in M_1 and $M_{2,0} = \cup_{n \geq 1} M_{2,n}$ is weakly dense in M_2 . Assume that $u \in M_{1,1}$ and $v \in M_{2,1}$. Denote by M_n the algebra generated by $M_{1,n}$ and $M_{2,n}$. Then $M_0 = \cup_{n \geq 1} M_n$ is weakly dense in M and $\delta(M_n) \subset \text{span } M_n e_B M_n$, for all $n \geq 1$. Since u, v are unitaries, $u \in M_1 \ominus B$ and $v \in M_2 \ominus B$, it follows that $\|\delta((uv)^n)\|_2 = \sqrt{2n}$, for all $n \geq 1$. This shows that $\delta|_{M_0}$ is unbounded. Thus, δ satisfies all the assumptions required in Theorem 7.1.

We continue by verifying the rest of the assumptions from Theorem 7.1.

We first claim that M is a factor and does not have property Gamma. Let $x \in M' \cap M^\omega$. Since $E_B(u) = E_B(v) = E_B(w) = E_B(w^*v) = 0$, [Io12a, Lemma 6.1] gives that $M' \cap M^\omega \subset B^\omega$, and thus $x \in B^\omega$. Recall that $zBz^* \perp B$, for some $z \in \{u, v\}$. Therefore, $\langle z(x - \tau(x))z^*, x - \tau(x) \rangle = 0$. On the other hand, since x commutes with z , we have that $z(x - \tau(x))z^* = x - \tau(x)$. Altogether, it follows that $x = \tau(x) \in \mathbb{C}1$, thereby proving the claim.

Next, using the same argument as in the proof of [Io12a, Lemma 6.1], we prove that $S = \{u, v, w\}$ is a non-amenability set for M relative to B . For $i \in \{1, 2\}$, we denote by $W_i \subset M$ the set of alternating words in $M_1 \ominus B$ and $M_2 \ominus B$ which start in $M_i \ominus B$, and by $\mathcal{H}_i \subset L^2(\langle M, e_B \rangle)$ the $\|\cdot\|_2$ -closure of the linear span of $W_i e_B M$. Let $\mathcal{H}_0 \subset L^2(\langle M, e_B \rangle)$ be the $\|\cdot\|_2$ -closure of $e_B M$. Then $L^2(\langle M, e_B \rangle) = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. For $i \in \{0, 1, 2\}$, let e_i be the orthogonal projection onto \mathcal{H}_i .

Notice that if $x \in M_1 \ominus B$ and $y \in M_2 \ominus B$ then $x\mathcal{H}_2x^* \subset \mathcal{H}_1$ and $y\mathcal{H}_1y^* \subset \mathcal{H}_2$. Since $E_B(u) = E_B(v) = E_B(w) = E_B(w^*v) = 0$, we deduce that

$$(7.1) \quad u\mathcal{H}_2u^* \subset \mathcal{H}_1, \quad v\mathcal{H}_1v^* \subset \mathcal{H}_2, \quad w\mathcal{H}_1w^* \subset \mathcal{H}_2 \quad \text{and} \quad v\mathcal{H}_1v^* \perp w\mathcal{H}_1w^*.$$

Now, let $\xi \in L^2(\langle M, e_B \rangle)$ and denote $C_\xi = \sum_{z \in S} \|z\xi z^* - \xi\|_2$. Then equation 7.1 implies that

$$\|e_2(u^*\xi u)\|_2 \leq \|e_1(\xi)\|_2 \quad \text{and} \quad \|e_1(v^*\xi v)\|_2^2 + \|e_1(w^*\xi w)\|_2^2 \leq \|e_2(\xi)\|_2^2.$$

These inequalities further imply that $\|e_2(\xi)\|_2 - C_\xi \leq \|e_1(\xi)\|_2$ and $\sqrt{2}(\|e_1(\xi)\|_2 - C_\xi) \leq \|e_2(\xi)\|_2$. From this it is easy to derive that $\|e_1(\xi)\|_2 \leq 6C_\xi$ and $\|e_2(\xi)\|_2 \leq 7C_\xi$. Since $u\mathcal{H}_0u^* \subset \mathcal{H}_1$, we similarly get that $\|e_0(\xi)\|_2 \leq \|e_1(\xi)\|_2 + C_\xi \leq 7C_\xi$. Altogether, it follows that $\|\xi\|_2 \leq 20C_\xi$, proving that S is indeed a non-amenability set relative to B .

Finally, let $r \in M$ be a non-zero projection. We claim that there exists a mixing B - B bimodule $\mathcal{H} \subset L^2(M)$ such that $r\mathcal{H}r \neq \{0\}$. Assume by contradiction that this is false. Let $z_1, z_2 \in \{u, v\}$ such that $z_1Bz_1^* \perp B$ and $\{z_1, z_2\} = \{u, v\}$. Denote $z = z_2z_1$. Then for every $k \geq 1$, by using freeness, it is clear that $z^k B z^{k*} \perp B$. Thus, the B - B bimodule $\mathcal{H}_k = \overline{Bz^k B}^{\|\cdot\|_2}$ is isomorphic to the coarse B - B bimodule, $L^2(B) \bar{\otimes} L^2(B)$, and is therefore mixing.

By our assumption we have that $r\mathcal{H}_k r = \{0\}$, hence $rz^k r = 0$, for all $k \geq 1$. In particular, we get that $(\frac{1}{n} \sum_{k=1}^n z^{k*} r z^k) r = 0$, for all $n \geq 1$. Let $D \subset M$ denote the von Neumann subalgebra generated by z . Von Neumann's ergodic theorem implies that $\frac{1}{n} \sum_{k=1}^n z^{k*} r z^k$ converges in $\|\cdot\|_2$ to $E_{D' \cap M}(r)$, as $n \rightarrow \infty$. Thus, we derive that $E_{D' \cap M}(r)r = 0$ and further that $(E_{D' \cap M}(r))^2 = 0$. Since $r \geq 0$ it follows that $E_{D' \cap M}(r) = 0$ and $\tau(r) = \tau(E_{D' \cap M}(r)) = 0$, hence $r = 0$. This provides the desired contradiction.

Altogether, we can apply Theorem 7.1 and derive the conclusion. \square

In the proof of Theorem 7.1 we will need the following technical result which says that if $L^2(M)$ contains “enough” mixing B - B bimodules, then no Cartan subalgebra of M can be embedded into B . More precisely, we have

Proposition 7.2. *Let M be a II_1 factor and $B \subset M$ be a von Neumann subalgebra. Assume that for any non-zero projection $r \in B' \cap M$, there exists a mixing B - B sub-bimodule \mathcal{H} of $L^2(M)$ such that $r\mathcal{H}r \neq \{0\}$.*

- (1) *If $A \subset M$ is a Cartan subalgebra, then $A \not\prec_M B$.*
- (2) *If M has property Γ , then $M' \cap M^\omega \not\prec_{M^\omega} B^\omega$. Moreover, in this case, let (N, τ') be any tracial von Neumann algebra containing B such that $\tau|_B = \tau'|_B$ and denote $\tilde{M} = M *_B N$. Then $M' \cap M^\omega \not\prec_{\tilde{M}^\omega} B^\omega$.*

Proof. (1) Assume by contradiction that $A \subset M$ is a Cartan subalgebra such that $A \prec_M B$. Then we can find projections $p \in A, q \in B$, a non-zero partial isometry $v \in M$ and a $*$ -homomorphism $\phi : Ap \rightarrow qBq$ such that $v^*v = p$, $q_0 = vv^* \leq q$ and $\phi(x)v = vx$, for all $x \in Ap$.

Towards a contradiction, let $r \in B' \cap M$ be the smallest projection such that $q_0 \leq r$. By the hypothesis we can find a mixing B - B bimodule $\mathcal{H} \subset L^2(M)$ such that $r\mathcal{H}r \neq \{0\}$. Denote by e the orthogonal projection from $L^2(M)$ onto \mathcal{H} .

Denote $B_0 = \phi(Ap) \subset qBq$. Fix $u \in \mathcal{N}_{pMp}(Ap)$ and denote by θ the automorphism of B_0 given by $\theta = \phi \circ \text{Ad}(u) \circ \phi^{-1}$. Then we have that

$$(7.2) \quad vuv^*y = \theta(y)vuv^* \quad \text{for all } y \in B_0$$

Since B_0 is diffuse, we can find a sequence $y_n \in \mathcal{U}(B_0)$ such that $y_n \rightarrow 0$, weakly. Since \mathcal{H} is a B - B bimodule by using equation 7.2 we get that $e(vuv^*)y_n = \theta(y_n)e(vuv^*)$, for all n . Hence we have that $\langle \theta(y_n^*)e(vuv^*)y_n, e(vuv^*) \rangle = \|e(vuv^*)\|_2^2$, for all n . Since $y_n \rightarrow 0$ weakly and \mathcal{H} is a mixing B - B bimodule, we derive that $e(vuv^*) = 0$.

Since this holds for any unitary $u \in \mathcal{N}_{pMp}(Ap)$ and $Ap \subset pMp$ is a Cartan subalgebra, we conclude that $e(q_0Mq_0) = e(vpMp v^*) = \{0\}$. Let us show that this implies that $e(rMr) = \{0\}$.

Indeed, denote by \mathcal{K} the $\|\cdot\|_2$ closure of the convex hull of the set $\{wq_0w^* | w \in \mathcal{U}(B)\}$. Since $e(q_0Mq_0) = \{0\}$ and e is the orthogonal projection onto a B - B bimodule, it follows that we have $e(zMz) = \{0\}$, for all $z \in \mathcal{K}$. Since $E_{B' \cap M}(q_0) \in \mathcal{K}$ (more precisely, $E_{B' \cap M}(q_0)$ is the unique element of minimal $\|\cdot\|_2$ in \mathcal{K}) we get that $e(E_{B' \cap M}(q_0)ME_{B' \cap M}(q_0)) = \{0\}$. Since r is equal to the support of $E_{B' \cap M}(q_0)$, we have that $E_{B' \cap M}(q_0)ME_{B' \cap M}(q_0)$ is a $\|\cdot\|_2$ -dense subspace of rMr . Thus, we would get that $e(rMr) = \{0\}$ or, equivalently, that $r\mathcal{H}r = \{0\}$, which provides the desired contradiction.

- (2) First, assume by contradiction that M has property Gamma and that $M' \cap M^\omega \prec_{M^\omega} B^\omega$.

Let $\{y_i\}_{i \geq 1}$ be a $\|\cdot\|_2$ dense sequence in M . Since M has property Gamma, by a construction of Popa (see the proof of [Oz03, Proposition 7]) we can find diffuse abelian von Neumann subalgebras $\{A_n\}_{n \geq 1}$ of M such that for all n we have that $A_{n+1} \subset A_n$ and that

$$(7.3) \quad \|y_i - E_{A_n \cap M}(y_i)\|_2 \leq \frac{1}{n}, \quad \text{for all } 1 \leq i \leq n.$$

Then we have

Claim. $A_n \prec_M B$, for some $n \geq 1$.

Proof of the claim. Denote by $A_\omega = \prod_{n=1}^\omega A_n$ the von Neumann subalgebra of M^ω consisting of all $x = (x_n)_n$ such that $\lim_{n \rightarrow \omega} \|x_n - E_{A_n}(x_n)\|_2 = 0$.

Then 7.3 implies that $A_\omega \subset M' \cap M^\omega$. Since $M' \cap M^\omega \prec_{M^\omega} B^\omega$, we get that $A_\omega \prec_{M^\omega} B^\omega$. Thus, we can find projections $p \in A_\omega, q \in B^\omega$, a non-zero partial isometry $v \in qM^\omega r$ and a $*$ -homomorphism $\phi : A_\omega p \rightarrow qB^\omega q$ such that $\phi(x)v = vx$, for all $x \in A_\omega r$.

Let $\delta = \|E_{B^\omega}(vv^*)\|_2$. Then $\|E_{B^\omega}(vuv^*)\|_2 = \delta$, for all $u \in \mathcal{U}(A_\omega p)$. Write $p = (p_n)_n$ and $v = (v_n)_n$, where $p_n \in A_n$ is a projection and $v_n \in M$, for all n .

If the claim is false, then $A_n \not\prec_M B$ and thus $A_n p_n \not\prec_M B$, for all $n \geq 1$. Thus, for every $n \geq 1$, we can find a unitary $u_n \in A_n p_n$ such that $\|E_B(v_n u_n v_n^*)\|_2 \leq \frac{\delta}{2}$. Then the unitary $u = (u_n) \in A_\omega p$ satisfies $\|E_{B^\omega}(vuv^*)\|_2 = \lim_{n \rightarrow \omega} \|E_B(v_n u_n v_n^*)\|_2 \leq \frac{\delta}{2}$, which gives a contradiction. \square

Let n such that $A_n \prec_M B$. Then we can find projections $p \in A_n, q \in B$, a non-zero partial isometry $v \in qMp$ and a $*$ -homomorphism $\phi : A_n p \rightarrow qBq$ such that $\phi(x)v = vx$, for all $x \in A_n p$. Denote $q_0 = vv^* \leq q$ and let $r \in B' \cap M$ be the smallest projection such that $q_0 \leq r$. The hypothesis implies the existence of a non-zero mixing B - B bimodule $\mathcal{H} \subset L^2(M)$ such that $r\mathcal{H}r \neq \{0\}$. Denote by e the orthogonal projection from $L^2(M)$ onto \mathcal{H} .

Now, let $N \geq n$, $u \in A'_N \cap M$ and $x \in A_N p$. Write $x = x_0 p$, where $x_0 \in A_N$. Since u and x_0 commute and $v = vp$ we get that $vuv^* \phi(x) = vuv^* x = vx_0 v^* = vx_0 u v^* = vxuv^* = \phi(x) vuv^*$. This shows that $v(A'_N \cap M)v^* \subset q_0 M q_0$ commutes with $\phi(A_N p) \subset qBq$.

Since \mathcal{H} is a mixing B - B bimodule and A_N is diffuse, by repeating the argument from the proof of (1) we get that $e(v(A'_N \cap M)v^*) = \{0\}$, for all $N \geq n$. Equation 7.3 then implies that $e(vy_i v^*) = 0$, for all $i \geq 1$. By using the $\|\cdot\|_2$ density of $\{y_i\}_{i \geq 1}$ in M we conclude that $e(q_0 M q_0) = e(vMv^*) = \{0\}$ and the end of the proof of (1) yields a contradiction.

To prove the moreover assertion, assume by contradiction that $M' \cap M^\omega \prec_{\tilde{M}^\omega} B^\omega$. Then the above claim implies that $A_n \prec_{\tilde{M}} B$, for some $n \geq 1$. Since $A_n \subset M$ and $\tilde{M} = M *_B N$, by [IPP05, Theorem 1.2.1] we get that $A_n \prec_M B$. Continuing as above yields a contradiction. \square

7.2. Proof of Theorem 7.1. Define $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$ and let $\{\alpha_t\}_{t \in \mathbb{R}}$ be the one-parameter group of automorphisms of \tilde{M} arising from δ as provided by Proposition 3.3. Note that since $\delta|_B \equiv 0$, we have that $\alpha_t(x) = x$, for all $x \in B$ and every $t \in \mathbb{R}$. Let $S \subset M_0$ be a non-amenability set for M relative to B .

The core of the proof consists of proving several claims about the inclusion $M \subset \tilde{M}$.

Claim 1. $\delta|_{M_0}$ is unbounded.

Proof of Claim 1. Let $s \in L(\mathbb{Z})$ be a semicircular element and $L^2(\langle M, e_B \rangle) \ni \xi \rightarrow \xi \# s \in L^2(\tilde{M})$ be the unique embedding of M - M bimodules sending e_B to s . Let $a, b, c \in M_0$. Since δ is a real derivation, a calculation in the spirit of the proof of [Vo98, Proposition 4.1] gives that

$$\begin{aligned} & \langle a\delta^*(e_B)b - E_M((\delta(a)\#s)s)b - aE_M(s(\delta(b)\#s)), c \rangle = \\ & \quad \langle \delta^*(e_B), a^*cb^* \rangle - \langle \delta(a)\#s, cb^*s \rangle - \langle \delta(b)\#s, sa^*c \rangle = \\ & \quad \langle \delta(bc^*a) - bc^*\delta(a) - \delta(b)c^*a, e_B \rangle = \langle b\delta(c^*)a, e_B \rangle = \langle ae_Bb, \delta(c) \rangle. \end{aligned}$$

Thus, ae_Bb belongs to the domain of δ^* , for all $a, b \in M_0$. Hence, for all $x \in D(\delta)$, we have that

$$\|\delta(x)\|_2 = \sup_{y \in \text{span}(M_0 e_B M_0), \|y\|_2 \leq 1} |\langle \delta(x), y \rangle| = \sup_{y \in \text{span}(M_0 e_B M_0), \|y\|_2 \leq 1} |\langle x, \delta^*(y) \rangle|$$

Since δ is unbounded and $M_0 \subset D(\delta)$ is dense in $\|\cdot\|_2$, it follows that $\delta|_{M_0}$ is unbounded. \square

Claim 2. $M' \cap \tilde{M}^\omega \subset M^\omega$.

Proof of Claim 2. Since $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$, there exists a B - M bimodule \mathcal{K} such that as M - M bimodules we have $L^2(\tilde{M}) \ominus L^2(M) \cong L^2(M) \otimes_B \mathcal{K}$. Since S is a non-amenability

set for M relative to B , the second part of Lemma 2.6 implies that there is $\kappa > 0$ such that $\|\xi\|_2 \leq \kappa \sum_{y \in S} \|y\xi - \xi y\|_2$, for all $\xi \in L^2(\tilde{M}) \ominus L^2(M)$. This gives that $M' \cap \tilde{M}^\omega \subset M^\omega$. \square

Note that since M is a factor, Claim 2 implies that $M' \cap \tilde{M} = \mathbb{C}1$.

Claim 3. $\alpha_t(M)$ is not amenable relative to B inside \tilde{M} , and $\alpha_t(M) \not\prec_{\tilde{M}} B \bar{\otimes} L(\mathbb{Z})$, for any $t \in \mathbb{R}$.

Proof of Claim 3. Consider the B - M bimodule $\mathcal{H} = \mathcal{K} \otimes_B L^2(\tilde{M})$, where \mathcal{K} is as in the proof of Claim 2. Then we have that $L^2(\langle \tilde{M}, e_B \rangle) \cong L^2(\tilde{M}) \otimes_B L^2(\tilde{M}) \cong L^2(M) \otimes_B (L^2(\tilde{M}) \oplus \mathcal{H})$, as M - M bimodules. The second part of Lemma 2.6 now implies that S is a non-amenability set for M relative to B inside \tilde{M} . In particular, M is not amenable relative to B inside \tilde{M} . Since α_t leaves B invariant, we derive that $\alpha_t(M)$ is not amenable relative to B inside \tilde{M} , for any $t \in \mathbb{R}$.

Assume by contradiction that $\alpha_t(M) \prec_{\tilde{M}} B \bar{\otimes} L(\mathbb{Z})$, for some $t \in \mathbb{R}$. Since $\alpha_t(M)' \cap \tilde{M} = \mathbb{C}1$, by [Io12a, Remark 2.2] it follows that $\alpha_t(M)$ is amenable relative to $B \bar{\otimes} L(\mathbb{Z})$ inside \tilde{M} . Note that $B \bar{\otimes} L(\mathbb{Z})$ is amenable relative to B inside \tilde{M} . Indeed, if $u \in L(\mathbb{Z})$ is a generating Haar unitary, then the vectors $\xi_n = \frac{1}{\sqrt{n}} \sum_{k=1}^n u^k e_B u^{k*} \in L^2(\langle \tilde{M}, e_B \rangle)$ satisfy $\langle x\xi_n, \xi_n \rangle = \tau(x)$, for all $x \in \tilde{M}$, and $\|y\xi_n - \xi_n y\|_2 \rightarrow 0$, for all $y \in B \bar{\otimes} L(\mathbb{Z})$.

By combining the last two facts and using [OP07, Proposition 2.4 (3)] we deduce that $\alpha_t(M)$ is amenable relative to B inside \tilde{M} . This leads to a contradiction. \square

Claim 4. There exists $t_0 > 0$ such that $\alpha_t(M) \not\prec_{\tilde{M}} M$, for all $t \in (0, t_0)$.

Proof of Claim 4. Assuming that the claim is false, we can find a sequence $t_n \rightarrow 0$ with $t_n > 0$ such that $\alpha_{t_n}(M) \prec_{\tilde{M}} M$, for all $n \geq 1$. On the other hand, Claim 3 gives that $\alpha_{t_n}(M) \not\prec_{\tilde{M}} B$. Recall that $\alpha_{t_n}(M)' \cap \tilde{M} = \mathbb{C}1$, M is a factor and $\tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$. By combining all these facts, the proof of [IPP05, Theorem 5.1] implies that we can find a unitary operator $v_n \in \tilde{M}$ such that $v_n \alpha_{t_n}(M) v_n^* \subset M$.

Since S is a non-amenability set for M relative to B inside \tilde{M} , Lemma 2.7 (2) provides a constant $C > 0$ such that for every $n \geq 1$ we have

$$\|\alpha_{t_n}(x) - E_M(\alpha_{t_n}(x))\|_2 \leq C \sum_{y \in S} \|\alpha_{t_n}(y) - y\|_2, \text{ for all } x \in (M)_1.$$

Finally, let $x \in M_0$. Proposition 3.3 gives that $\|\frac{\alpha_t(x) - x}{t} - \delta(x) \# s\|_2 \rightarrow 0$, as $t \rightarrow 0$. Also, we have that $\|\delta(x) \# s\|_2 = \|\delta(x)\|_2$ and $E_M(\delta(x) \# s) = 0$. By combining these facts with the last inequality we get that $\|\delta(x)\|_2 \leq C \sum_{y \in S} \|\delta(y)\|_2$, for all $x \in M_0$, which contradicts Claim 1. \square

Claim 5. M does not have property Gamma.

Proof of Claim 5. Assume by contradiction that M has property Gamma and let $t \in (0, t_0)$, where t_0 is given by Claim 4. Proposition 7.2 (2) then implies that $M' \cap M^\omega \not\prec_{\tilde{M}^\omega} B^\omega$. Since B is invariant under α_t , we get that $\alpha_t(M)' \cap \tilde{M}^\omega \not\prec_{\tilde{M}^\omega} B^\omega$. Also, by the above claims we have that $\alpha_t(M) \not\prec_{\tilde{M}} B \bar{\otimes} L(\mathbb{Z})$ and $\alpha_t(M) \not\prec_{\tilde{M}} M$.

By applying [Io12a, Theorem 6.3] to the inclusion $\alpha_t(M) \subset \tilde{M} = M *_B (B \bar{\otimes} L(\mathbb{Z}))$, we deduce that $\alpha_t(M)p$ is amenable relative to B inside \tilde{M} , for a non-zero projection $p \in \alpha_t(M)' \cap \tilde{M}$. Since $\alpha_t(M)' \cap \tilde{M} = \mathbb{C}1$, this would imply that $\alpha_t(M)$ is amenable relative to B inside \tilde{M} , which is false, by Claim 3. \square

We are now ready to prove the conclusion of Theorem 7.1. Thus, assume by contradiction that M has a Cartan subalgebra A . Let $t \in (0, t_0)$. Note that $\alpha_t(M) \subset \mathcal{N}_{\tilde{M}}(\alpha_t(A))''$ and therefore $\alpha_t(M) \subset \mathcal{N}_{\tilde{M}}(\alpha_t(A))''$. By combining Claims 2 and 5, we get that $M' \cap \tilde{M}^\omega = \mathbb{C}1$. Thus, we also

have that $\alpha_t(M)' \cap \tilde{M}^\omega = \mathbb{C}1$. Altogether, we are in position to apply Theorem 2.12 (in the case $Q = \mathbb{C}1$) and deduce that one of the following conditions holds:

- (1) $\alpha_t(A) \prec_{\tilde{M}} B$.
- (2) $\alpha_t(M) \prec_{\tilde{M}} M$.
- (3) $\alpha_t(M) \prec_{\tilde{M}} B \bar{\otimes} L(\mathbb{Z})$.
- (4) $\alpha_t(M)$ is amenable relative to B inside \tilde{M} .

Since α_t leaves B invariant, condition (1) implies that $A \prec_{\tilde{M}} B$. Since $A \subset M$, [IPP05, Theorem 1.2.1] implies that $A \prec_M B$. This however cannot happen by Proposition 7.2 (1). Since conditions (2)-(4) are also false as shown above, we get a contradiction. \square

8. ALGEBRAIC COCYCLES AND UNIQUENESS OF CARTAN SUBALGEBRAS

In this final section we first prove a slightly more general form of Theorem 1.6 and then derive Corollary 1.7.

Theorem 8.1. *Let Γ be a group satisfying all the assumptions from Theorem 1.6. Assume that Γ' is a group which admits a finite normal subgroup N such that $\Gamma'/N \cong \Gamma$.*

Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma'$, up to unitary conjugacy, for any free ergodic probability measure preserving action $\Gamma' \curvearrowright (X, \mu)$.

Proof. Let $\Lambda < \Gamma$ be a subgroup and $b : \Gamma \rightarrow \mathbb{C}(\Gamma/\Lambda)$ a cocycle satisfying the hypothesis of Theorem 1.6. After replacing b with its real or imaginary part, we may assume that we have $b(\Gamma) \subset \mathbb{R}(\Gamma/\Lambda)$. For $g \in \Gamma$, write $b(g) = \sum_{h\Lambda \in \Gamma/\Lambda} c_{g,h\Lambda} \delta_{h\Lambda}$, where $c_{g,h\Lambda} \in \mathbb{R}$.

Consider the isometry $V : \ell^2(\Gamma/\Lambda) \rightarrow L^2(\langle L(\Gamma), e_{L(\Lambda)} \rangle)$ given by $V(\delta_{h\Lambda}) = u_h e_{L(\Lambda)} u_h^*$. Then $V(\pi(g)\xi) = u_g \xi u_g^*$, where $\pi : \Gamma \rightarrow \ell^2(\Gamma/\Lambda)$ is the quasi-regular representation $\pi(g)(\delta_{h\Lambda}) = \delta_{gh\Lambda}$.

We define $\delta : \mathbb{C}\Gamma \rightarrow L^2(\langle L(\Gamma), e_{L(\Lambda)} \rangle)$ by putting $\delta(u_g) = i V(b(g)) u_g^*$, or, explicitly,

$$\delta(u_g) = i \sum_{h\Lambda \in \Gamma/\Lambda} c_{g,h\Lambda} u_h e_{L(\Lambda)} u_h^* u_g, \text{ for all } g \in \Gamma.$$

Then it is easy to see that δ is a real derivation. Since $b|_\Lambda \equiv 0$, we have that $\delta|_{\mathbb{C}\Lambda} \equiv 0$. As for every $g \in \Gamma$ we have that $\text{Tr}(\delta(u_g) e_{L(\Lambda)}) = \delta_{g,e} c_{e,e\Lambda} = \delta_{g,e} \langle b(e), \delta_{e\Lambda} \rangle = 0$, it follows that $\delta^*(e_{L(\Lambda)}) = 0$. Since $b(\Gamma_n) \subset \mathbb{C}(\Gamma_n \Lambda / \Lambda)$, we get that $\delta(\mathbb{C}\Gamma_n) \subset \text{span}(\mathbb{C}\Gamma_n e_{L(\Lambda)} \mathbb{C}\Gamma_n)$, for all $n \geq 1$. Since $\mathbb{C}\Gamma_n$ is finitely generated and $\mathbb{C}\Gamma = \cup_{n \geq 1} \mathbb{C}\Gamma_n$, we are in position to apply Proposition 3.3.

Let $\tilde{\Gamma} = \Gamma *_\Lambda (\Lambda \times \mathbb{Z})$ and $s \in L(\mathbb{Z})$ a generating semicircular element. Proposition 3.3 provides a one-parameter group of automorphisms $\{\alpha_t\}_{t \in \mathbb{R}}$ of $L(\tilde{\Gamma})$ satisfying $\|\frac{\alpha_t(u_g) - u_g}{t} - \delta(u_g) \# s\|_2 \rightarrow 0$, for all $g \in \Gamma$. Since $\delta|_{\mathbb{C}\Lambda} \equiv 0$ and $\delta^*(e_{L(\Lambda)}) = 0$, we have that $\alpha_t(x) = x$, for all $x \in L(\Lambda \times \mathbb{Z})$.

For $f \in \ell^\infty(\Gamma/\Lambda)$ and $h \in \Gamma$, we define $\sigma(h)(f) \in \ell^\infty(\Gamma/\Lambda)$ by letting $\sigma(h)(f)(g\Lambda) = f(h^{-1}g\Lambda)$. Since Λ is not co-amenable in Γ , there exists a finite set $S \subset \Gamma$ such that there is no $\sigma(S)$ -invariant state on $\ell^\infty(\Gamma/\Lambda)$. Consider the unital $*$ -homomorphism $\rho : \ell^\infty(\Gamma/\Lambda) \rightarrow \langle L(\Gamma), e_{L(\Lambda)} \rangle$ given by $\rho(f) = \sum_{g\Lambda \in \Gamma/\Lambda} f(g\Lambda) u_g e_{L(\Lambda)} u_g^*$. Then $\rho(\sigma(h)(f)) = u_h \rho(f) u_h^*$, for all $h \in \Gamma$. Thus, there is no S -central state on $\langle L(\Gamma), e_{L(\Lambda)} \rangle$ and hence S is a non-amenable set for $L(\Gamma)$ relative to $L(\Lambda)$.

Since δ is unbounded and $S \subset \Gamma$ is a non-amenable set for $L(\Gamma)$ relative to $L(\Lambda)$, Claims 1-3 from the proof of Theorem 7.1 (applied here verbatim in the case $M = L(\Gamma)$ and $B = L(\Lambda)$) give the following:

- $L(\Gamma)' \cap L(\tilde{\Gamma})^\omega \subset L(\Gamma)^\omega$.

- $\alpha_t(L(\Gamma))$ is not amenable relative to $L(\Lambda)$ inside $L(\tilde{\Gamma})$, for any $t \in \mathbb{R}$.
- $\alpha_t(\Gamma) \not\prec_{L(\tilde{\Gamma})} L(\Lambda \times \mathbb{Z})$, for any $t \in \mathbb{R}$.
- there exists $t_0 > 0$ such that $\alpha_t(L(\Gamma)) \not\prec_{L(\tilde{\Gamma})} L(\Gamma)$, for any $t \in (0, t_0)$.

Now, let $\Gamma' \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Define $M = L^\infty(X) \rtimes \Gamma'$ and let A be a Cartan subalgebra of M . We want to show that A is unitarily conjugate to $L^\infty(X)$.

To this end, let $\Delta : M \rightarrow M \bar{\otimes} L(\Gamma)$ be the $*$ -homomorphism given by $\Delta(au_g) = au_g \otimes u_{p(g)}$, for all $a \in L^\infty(X)$ and $g \in \Gamma'$ [PV09]. Here, $p : \Gamma' \rightarrow \Gamma$ denotes the quotient homomorphism. Further, we fix $t \in (0, t_0)$ and define $\theta_t = (\text{id}_M \otimes \alpha_t) \circ \Delta : M \rightarrow M \bar{\otimes} L(\tilde{\Gamma})$.

Then $\theta_t(M) \subset \mathcal{N}_{M \bar{\otimes} L(\tilde{\Gamma})}(\theta_t(A))''$, therefore $u_g \otimes \alpha_t(u_{p(g)}) \in \mathcal{N}_{M \bar{\otimes} L(\tilde{\Gamma})}(\theta_t(A))''$, for all $g \in \Gamma$. Since $L(\Gamma)' \cap L(\tilde{\Gamma})^\omega \subset L(\Gamma)^\omega$ and $L(\Gamma)$ does not have property Gamma, we have $\alpha_t(L(\Gamma))' \cap L(\tilde{\Gamma})^\omega = \mathbb{C}1$. By applying Theorem 2.12 we conclude that one of the following conditions holds:

- (1) $\theta_t(A) \prec_{M \bar{\otimes} L(\tilde{\Gamma})} M \bar{\otimes} L(\Lambda)$.
- (2) $\alpha_t(L(\Gamma)) \prec_{L(\tilde{\Gamma})} L(\Gamma)$.
- (3) $\alpha_t(L(\Gamma)) \prec_{L(\tilde{\Gamma})} L(\Lambda \times \mathbb{Z})$.
- (4) $\alpha_t(L(\Gamma))$ is amenable relative to $L(\Lambda)$ inside $L(\tilde{\Gamma})$.

Since conditions (2)-(4) cannot hold by the above, condition (1) must be true. Since α_t leaves $L(\Lambda)$ invariant, (1) is equivalent to having $\Delta(A) \prec_{M \bar{\otimes} L(\tilde{\Gamma})} M \bar{\otimes} L(\Lambda)$. Since $\Delta(A) \subset M \bar{\otimes} L(\Gamma)$ and $L(\tilde{\Gamma}) = L(\Gamma) *_{L(\Lambda)} L(\Lambda \times \mathbb{Z})$, [IPP05, Theorem 1.2.1] gives that $\Delta(A) \prec_{M \bar{\otimes} L(\Gamma)} M \bar{\otimes} L(\Lambda)$.

By using [Io12a, Lemma 7.2] we derive that $A \prec_M L^\infty(X) \rtimes p^{-1}(\Lambda)$. For $i \in \{1, 2, \dots, m\}$, let $g'_i \in \Gamma'$ such that $p(g'_i) = g_i$. Since M is a factor, [HPV10, Proposition 8] implies that $A \prec_M L^\infty(X) \rtimes (\cap_{i=1}^m g'_i p^{-1}(\Lambda) g_i'^{-1})$. Since $\cap_{i=1}^m g_i \Lambda g_i^{-1}$ is finite, $\cap_{i=1}^m g'_i p^{-1}(\Lambda) g_i'^{-1}$ is also finite. By combining the last two facts we get that $A \prec_M L^\infty(X)$. Since A and $L^\infty(X)$ are Cartan subalgebras of M , [Po01, Theorem A.1] yields that they are unitarily conjugate (see also [Va06, Theorem C.3]). \square

Turning to the proof of Corollary 1.7, let us first establish the following technical result.

Lemma 8.2. *Let G be a countable group, $\Lambda < G$ be a subgroup, and $\theta : \Lambda \rightarrow G$ be an injective group homomorphism. Assume that $\Lambda \neq G$ and $\theta(\Lambda) \neq G$. Denote by $\Gamma = \text{HNN}(G, \Lambda, \theta)$ the corresponding HNN extension. Then we have*

- (1) *If $\cap_{i=1}^m h_i \Lambda h_i^* = \{e\}$, for some $h_1, h_2, \dots, h_m \in \Gamma$, then Γ is not inner amenable.*
- (2) *Λ is not co-amenable in Γ .*

Remark 8.3. If $\Lambda = G$ or $\theta(\Lambda) = G$, then Λ is co-amenable in Γ by [MP03, Proposition 2].

Proof. Recall that $\Gamma = \langle G, t \mid t\lambda t^{-1} = \theta(\lambda), \forall \lambda \in \Lambda \rangle$, where t is the so-called stable letter. Let $A \subset G$ and $B \subset G$ be sets of representatives of the left cosets of Λ and $\theta(\Lambda)$ in G , respectively. We assume that $e \in A \cap B$. Since $\Lambda \neq G \neq \theta(\Lambda)$, we can find $a \in A \setminus \{e\}$ and $b \in B \setminus \{e\}$.

Below, we will implicitly use the *normal form theorem* [LS77, Chapter IV, Theorem 2.1]: every $g \in \Gamma$ can be uniquely written as a product $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0$, for some $g_0, g_1, \dots, g_n \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that $g_i \in A$, if $\varepsilon_i = -1$, and $g_i \in B$, if $\varepsilon_i = 1$, for all $i \in \{1, 2, \dots, n\}$, and that there is no consecutive subsequence $t^\varepsilon, 1, t^{-\varepsilon}$ within the sequence $g_n, t^{\varepsilon_n}, \dots, t^{\varepsilon_1}, g_0$.

(1) Assume by contradiction that Γ is inner amenable and let $\phi : \ell^\infty(\Gamma \setminus \{e\}) \rightarrow \mathbb{C}$ be a state which is invariant under the conjugation action of Γ . For a subset $S \subset \Gamma$, we denote $m(S) = \phi(1_{S \setminus \{e\}})$.

Denote by S the set of all $g = g_n t^{\varepsilon_n} g_{n-1} \dots g_1 t^{\varepsilon_1} g_0 \in \Gamma$ (represented in normal form) such that $n \geq 1$ and $g_n \neq e$. Also, denote by U (respectively, V) the set of $g \in \Gamma$ such that $n \geq 1$, $g_n = e$ and $\varepsilon_n = -1$ (respectively, $\varepsilon_n = 1$). Then we have that

$$t^{-1}St \subset U, \quad tSt^{-1} \subset V, \quad aUa^{-1} \subset S, \quad bVb^{-1} \subset S, \quad \text{and} \quad aUa^{-1} \cap bVb^{-1} = \emptyset.$$

Since m is a Γ -invariant finitely additive measure, these inclusions imply that $m(S) \leq m(U)$, $m(S) \leq m(V)$ and $m(U) + m(V) \leq m(S)$. From this we deduce that $m(S) = m(U) = m(V) = 0$. Since $S \cup U \cup V \cup G = \Gamma$ and $m(\Gamma) = 1$, we conclude that $m(G) = 1$.

Since $t^{-1}Gt \cap G = \Lambda$, we further get that $m(\Lambda) = 1$. Finally, since $\cap_{i=1}^m h_i \Lambda h_i^* = \{e\}$, we derive that $m(\{e\}) = 1$. This contradicts the fact that $m(\{e\}) = 0$.

(2) Assume by contradiction that Λ is co-amenable inside Γ and let $\phi : \ell^\infty(\Gamma/\Lambda) \rightarrow \mathbb{C}$ be a Γ -invariant state. For a subset $S \subset \Gamma/\Lambda$, we denote $m(S) = \phi(1_S)$. Also, we denote by $\pi : \Gamma \rightarrow \Gamma/\Lambda$ the canonical projection, and still consider U, V, S as in (1).

Next, we have that $t^{-1}S \subset U, tS \subset V, aU \subset S$ and $bV \subset S$. Moreover, $a\pi(U) \cap b\pi(V) = \emptyset$. Since π is Γ -equivariant and m is a finitely additive Γ -invariant measure, it follows as above that $m(\pi(S)) = m(\pi(U)) = m(\pi(V)) = 0$. Since $\pi(S) \cup \pi(U) \cup \pi(V) \cup G/\Lambda = \Gamma/\Lambda$ and $m(\Gamma/\Lambda) = 1$, we conclude that $m(G/\Lambda) = 1$. Finally, since $tG/\Lambda \cap G/\Lambda = \emptyset$, we would get that $m(\emptyset) = 1$, which gives the desired contradiction. \square

8.1. Proof of Corollary 1.7. Let $\Gamma = \text{HNN}(G, \Lambda, \theta)$. Since $\cap_{i=1}^m g_i \Lambda g_i^{-1}$ is finite, it follows that $N = \cap_{g \in \Gamma} g \Lambda g^{-1}$ is a finite normal subgroup of Γ such that $N < \Lambda$. Moreover, we can find $h_1, h_2, \dots, h_n \in \Gamma$ such that $N = \cap_{j=1}^n h_j \Lambda h_j^{-1}$. Denote $\Gamma_0 = \Gamma/N$, $G_0 = G/N$, $\Lambda_0 = \Lambda/N$ and let $p : \Gamma \rightarrow \Gamma_0$ be the quotient homomorphism.

Since the stable letter t normalizes N , if $g \in \Lambda$ then $\theta(g) = tgt^{-1} \in N$ if and only if $g \in N$. Therefore, $\theta : \Lambda \rightarrow G$ descends to an injective group homomorphism $\theta_0 : \Lambda_0 \rightarrow G_0$. Moreover, we have that Γ_0 is naturally isomorphic to $\text{HNN}(G_0, \Lambda_0, \theta_0)$ and $\cap_{j=1}^n p(h_j) \Lambda_0 p(h_j)^{-1} = \{e\}$. Since $\Lambda_0 \neq G_0$ and $\theta_0(\Lambda_0) \neq G_0$, by Lemma 8.2 we get that Γ_0 is not inner amenable (hence $L(\Gamma_0)$ is a II_1 factor without property Gamma) and Λ_0 is not co-amenable inside Γ_0 .

Next, we define $b : \Gamma_0 = \text{HNN}(G_0, \Lambda_0, \theta_0) \rightarrow \mathbb{C}(\Gamma_0/\Lambda_0)$ by letting $b(g) = 0$, for all $g \in G_0$, and $b(t) = t\Lambda_0$, where $t \in \Gamma_0$ is the stable letter. Then b is an unbounded cocycle. To see that b is unbounded, just note that if $g \in G_0 \setminus \Lambda_0$, then $\|b((gt)^n)\|_2 = \sqrt{n}$, for all $n \geq 0$.

Finally, let $\{G_n\}_{n \geq 1}$ be a sequence of finitely generated subgroups of G_0 such that $G_0 = \cup_{n \geq 1} G_n$. Let $\Gamma_n < \Gamma_0$ be the subgroup generated by G_n and t . Then Γ_n is finitely generated, $\Gamma_n \subset \Gamma_{n+1}$ and $b(\Gamma_n) \subset \mathbb{C}(\Gamma_n \Lambda/\Lambda)$, for all n . Moreover, $\cup_{n \geq 1} \Gamma_n = \Gamma_0$. Altogether, we can apply Theorem 8.1 to get the conclusion. \square

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